

Problem of The Month for October

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Finding A Sum Of Consecutive Integers

Given an integer n , we want to find a sum of consecutive integers equal to n . This sum must have more than one summand. We additionally want to know *all* such sums for any particular n . Any two sums with a different set of summands are considered different sums. That is,

$$2 + 3 + 4 = 9$$

is different from

$$(-1) + 0 + 1 + 2 + 3 + 4 = 9$$

even though the summands in the first are a subset of the second. We also count the sums

$$1 + 2 + \cdots + m$$

and

$$0 + 1 + 2 + \cdots + m$$

to be two different sums.

We are also looking to answer the specific question, how many sums of consecutive integers equal 105, and what are they?

Odd Factor Sums

Consider the following sum of consecutive integers for n .

$$n = \sum_{i=-\beta}^{\beta} (a + i)$$

Note that any sum of an odd number of consecutive integers can be written this way.

We can show that a sum like this can be generated using almost any¹ odd factor of n and its cofactor. If we bring the a out of the summation, we have

$$\begin{aligned} n &= a(2\beta + 1) + \left(\sum_{i=-\beta}^{\beta} i \right) \\ &= a(2\beta + 1) \end{aligned}$$

¹We will see exactly which odd factors soon.

Now, we can find a sum of consecutive integers that equals n using any odd factor $b = 2\beta + 1$ of n and its cofactor, a . Notice that $\beta = \frac{b-1}{2}$. Since we must be able to derive a $\beta \geq 0$ from b , we must have $b \geq 1$.²

We can now define an **odd factor sum** for n to be,

$$S_n(b) = \sum_{i=-\beta}^{\beta} (a + i)$$

where b is an odd factor of n , $b \geq 1$, $\beta = \frac{b-1}{2}$, a is the cofactor of b , and $n \geq 0$.³

Sums Derivable From Odd Factor Sums

Each $S_n(b)$ has a corresponding sum with different but related summands also equal to n which can be derived from it. How we determine the summands of this corresponding sum depends on $a - \beta$, the first summand in $S_n(b)$.

If $a - \beta > 0$, then we have a case like $S_{18}(3)$,

$$5 + 6 + 7 = 18$$

Notice this sum can also be written as,

$$[(-4) + (-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4] + [5 + 6 + 7] = 18$$

This sum has an even number of summands, so we couldn't have found it with an odd factor sum.

Generally, if we let $\gamma = a - \beta - 1$, then we have

$$\left(\sum_{j=-\gamma}^{\gamma} j \right) + S_n(b)$$

This sum has an even number of summands,⁴ so we wouldn't have been able to find it with the odd factor sum alone, and it is clearly still equal to n . We can derive a single summation for this as follows,

²The sum, $n = \sum_{i=-\beta}^{\beta} (a + i)$, is valid if $\beta = 0$ ($b = 1$), but it has only one summand, n . This is not the kind of sum we are looking for, but we will need this sum to derive a different sum later.

³We will discuss negative values for n later.

⁴Each part has an odd number of summands and the sum of two odd numbers is even.

$$\begin{aligned}
S_n(b) &= \sum_{i=-\beta}^{\beta} (a+i) = \sum_{i=a-\beta}^{a+\beta} i \\
&= \left(\sum_{k=-(a-\beta-1)}^{(a-\beta-1)} k \right) + \left(\sum_{i=a-\beta}^{a+\beta} i \right) \\
&= \sum_{j=-(a-\beta-1)}^{a+\beta} j
\end{aligned}$$

We've noted that when $b = 1$, we get a sum which is only the single summand n . This is not a sum of the kind we are looking for. However, the sum produced by $S_n(1)$ can produce a valid sum with this first derivation method. Namely,

$$\left(\sum_{j=-(n-1)}^{n-1} j \right) + n$$

This will be important if we try to enumerate all the consecutive integer sums for any n .

If $a - \beta \leq 0$, then we have a case like $S_{18}(9)$,

$$(-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6 = 18$$

Notice this sum can also be written as,

$$3 + 4 + 5 + 6 = 18$$

This sum has an even number of summands, so again, we couldn't have found it with an odd factor sum.

Generally, we have,

$$\sum_{j=\beta-a+1}^{a+\beta} j$$

If we rewrite this sum,

$$\begin{aligned}
\sum_{j=\beta-a+1}^{a+\beta} j &= \sum_{k=0}^{(a+\beta)-(\beta-a+1)} [(\beta-a+1) + k] \\
&= \sum_{k=0}^{2a-1} [(\beta-a+1) + k]
\end{aligned}$$

it is easy to see it also has an even number of summands, so we couldn't have found it with an odd factor sum alone. We can derive this summation as follows.

$$\begin{aligned}
S_n(b) &= \sum_{i=-\beta}^{\beta} (a+i) = \sum_{j=a-\beta}^{a+\beta} j \\
&= \left(\sum_{k=a-\beta}^{-(a-\beta)=\beta-a} k \right) + \left(\sum_{j=\beta-a+1}^{a+\beta} j \right) \\
&= 0 + \left(\sum_{j=\beta-a+1}^{a+\beta} j \right) \\
&= \sum_{j=\beta-a+1}^{a+\beta} j
\end{aligned}$$

We can now define the corresponding second sum derived from $S_n(b)$ as

$$s_n(b) = \begin{cases} \sum_{j=-(a-\beta-1)}^{a+\beta} j & \text{if } a - \beta > 0 \\ \sum_{j=\beta-a+1}^{a+\beta} j & \text{if } a - \beta \leq 0 \end{cases}$$

We have shown that for all $S_n(b)$, we can derive a second sum, $s_n(b)$, with an even number of summands.

Completeness

It is clear that any sum with an odd number of summands can be represented by

$$S_n(b) = \sum_{i=-\beta}^{\beta} (a+i)$$

To ensure our description is complete, we need to show that any sum with an even number of summands is one of the corresponding sums, $s_n(b)$.

Any sum with an even number of summands where the first summand is negative and the last summand is positive has a corresponding sum with an odd number of summands because a number of terms on either side of zero will cancel each other out and the zero itself can be removed. We are removing an odd number of summands from an even number of summands, so we are left with an odd number of summands.

Any sum with an even number of summands where the first summand is positive has a corresponding sum with an odd number of summands. Let x be

the first summand. We can add summands from $-(x-1)$ to $(x-1)$ (which sum to zero) to the original summation. We are adding an odd number of summands to an even number of summands, so our new sum has an odd number of summands.⁵

The Number of Sums of Consecutive Integers

Let $\zeta^1(n)$ be the number of odd divisors for n . The total number of sums of consecutive integers for any integer n is

$$2\zeta^1(n) - 1$$

This number comes from the fact that each odd divisor of n has two sums of consecutive integers associated with it. However, $\zeta^1(n)$ includes the factor 1, and there is only one sum associated with this factor that we are interested in, so we must subtract one for the sum which is simply n itself as the only summand.

Note that $n = 0$ is a special case. Since $\zeta^1(0) = \infty$, there are an infinite sums of consecutive integers equal to zero. The following sum illustrates this quite well.

$$\sum_{i=-\alpha}^{\alpha} i = 0 \quad \text{for all } \alpha \geq 0$$

Sums for Negative Integers

Throughout this exposition, we have assumed non-negative values for n . All the work we have done is valid for negative values as well. If we have $n < 0$, we can simply find the sums for $-n$, then multiply those sums by -1. This will produce valid sums of negative integers that sum to n .

Sums for 105

The specific problem we wanted to solve was to find *all* the sums of consecutive integers that equal 105. These are easily found by finding all the odd divisors of 105, $\{1, 3, 5, 7, 15, 21, 35, 105\}$, and using these values for b in $S_{105}(b)$ and $s_{105}(b)$. These sums are enumerated on the following page. There are $2\zeta^1(105) - 1 = 2(8) - 1 = 15$ of them.

⁵This same technique can be used for sums with an even number of summands where the first summand is negative and the last summand is negative. (This would be a sum for a negative integer, which is discussed later.) We can simply multiply this whole sum through by -1, use the above technique, then multiply the new sum through by -1 a second time so it sums to the correct number.

$$\begin{aligned}
s_{105}(1) &= (-104) + (-103) + (-102) + (-101) + (-100) + (-99) + (-98) + (-97) + (-96) + (-95) + \\
&(-94) + (-93) + (-92) + (-91) + (-90) + (-89) + (-88) + (-87) + (-86) + (-85) + (-84) + \\
&(-83) + (-82) + (-81) + (-80) + (-79) + (-78) + (-77) + (-76) + (-75) + (-74) + (-73) + \\
&(-72) + (-71) + (-70) + (-69) + (-68) + (-67) + (-66) + (-65) + (-64) + (-63) + (-62) + \\
&(-61) + (-60) + (-59) + (-58) + (-57) + (-56) + (-55) + (-54) + (-53) + (-52) + (-51) + \\
&(-50) + (-49) + (-48) + (-47) + (-46) + (-45) + (-44) + (-43) + (-42) + (-41) + (-40) + \\
&(-39) + (-38) + (-37) + (-36) + (-35) + (-34) + (-33) + (-32) + (-31) + (-30) + (-29) + \\
&(-28) + (-27) + (-26) + (-25) + (-24) + (-23) + (-22) + (-21) + (-20) + (-19) + (-18) + \\
&(-17) + (-16) + (-15) + (-14) + (-13) + (-12) + (-11) + (-10) + (-9) + (-8) + (-7) + \\
&(-6) + (-5) + (-4) + (-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + \\
&12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 + \\
&31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39 + 40 + 41 + 42 + 43 + 44 + 45 + 46 + 47 + 48 + 49 + \\
&50 + 51 + 52 + 53 + 54 + 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62 + 63 + 64 + 65 + 66 + 67 + 68 + \\
&69 + 70 + 71 + 72 + 73 + 74 + 75 + 76 + 77 + 78 + 79 + 80 + 81 + 82 + 83 + 84 + 85 + 86 + 87 + \\
&88 + 89 + 90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 + 100 + 101 + 102 + 103 + 104 + 105 \\
S_{105}(3) &= 34 + 35 + 36 \\
s_{105}(3) &= (-33) + (-32) + (-31) + (-30) + (-29) + (-28) + (-27) + (-26) + (-25) + (-24) + \\
&(-23) + (-22) + (-21) + (-20) + (-19) + (-18) + (-17) + (-16) + (-15) + (-14) + \\
&(-13) + (-12) + (-11) + (-10) + (-9) + (-8) + (-7) + (-6) + (-5) + (-4) + (-3) + \\
&(-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + \\
&18 + 19 + 20 + 21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 + 31 + 32 + 33 + 34 + 35 + 36 \\
S_{105}(5) &= 19 + 20 + 21 + 22 + 23 \\
s_{105}(5) &= (-18) + (-17) + (-16) + (-15) + (-14) + (-13) + (-12) + (-11) + (-10) + (-9) + (-8) + \\
&(-7) + (-6) + (-5) + (-4) + (-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \\
&9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 + 23 \\
S_{105}(7) &= 12 + 13 + 14 + 15 + 16 + 17 + 18 \\
s_{105}(7) &= (-11) + (-10) + (-9) + (-8) + (-7) + (-6) + (-5) + (-4) + (-3) + (-2) + (-1) + 0 + \\
&1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 \\
S_{105}(15) &= 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 \\
s_{105}(15) &= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 \\
S_{105}(21) &= (-5) + (-4) + (-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 \\
s_{105}(21) &= 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 \\
S_{105}(35) &= (-14) + (-13) + (-12) + (-11) + (-10) + (-9) + (-8) + (-7) + (-6) + (-5) + (-4) + (-3) + \\
&(-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 \\
s_{105}(35) &= 15 + 16 + 17 + 18 + 19 + 20 \\
S_{105}(105) &= (-51) + (-50) + (-49) + (-48) + (-47) + (-46) + (-45) + (-44) + (-43) + (-42) + (-41) + \\
&(-40) + (-39) + (-38) + (-37) + (-36) + (-35) + (-34) + (-33) + (-32) + (-31) + (-30) + \\
&(-29) + (-28) + (-27) + (-26) + (-25) + (-24) + (-23) + (-22) + (-21) + (-20) + (-19) + \\
&(-18) + (-17) + (-16) + (-15) + (-14) + (-13) + (-12) + (-11) + (-10) + (-9) + (-8) + \\
&(-7) + (-6) + (-5) + (-4) + (-3) + (-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + \\
&12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 + 21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 + 31 + 32 + \\
&33 + 34 + 35 + 36 + 37 + 38 + 39 + 40 + 41 + 42 + 43 + 44 + 45 + 46 + 47 + 48 + 49 + 50 + 51 + 52 + 53 \\
s_{105}(105) &= 52 + 53
\end{aligned}$$