CHAPTER 2

Review

2.1 Definitions in Chapter 2

2.1 Set; Element; Member; Universal Set
2.2 Subset
2.3 Proper Subset
2.4 The Empty Set, ∅
2.5 Set Equality
2.6 Cardinality; Infinite Set
2.7 Complement
2.8 Intersection
2.9 Union
2.10 Difference
2.11 Disjoint
2.12 Union and Intersection of Multiple Sets
2.13 Partition
2.14 Cartesian Product
2.15 Power Set
2.16 Symmetric Difference
2.22 Statement
2.23 Tautology, Contradiction, Conditional
2.24 Logical Equivalence
2.25 Inference; Rule of Inference
2.26 Boolean Algebra
2.27 Boolean Expression
2.28 Zero Divisor
2.32 Symmetric Difference

2.2 Sample Exam Questions

The solutions to these sample exam questions can be found in Section 2.4.

1. Let \( A = \{a, c, d, e\} \), \( B = \{c, e, f\} \), \( C = \{a, b, c\} \), and let the universal set, \( U \), be \( \{a, b, c, d, e, f, g\} \).
   (a) What is \( A \cap B? \)
   (b) What is \( A - B? \)
   (c) What is \(|A|? \)
   (d) What is \(\overline{A}? \)
   (e) What is the power set \(\mathcal{P}(C)? \)

2. (a) Produce the truth table for the proposition \( P \lor (Q \to (P \land Q)) \). You must use the standard ordering for the truth values of \( P \) and \( Q \).
   (b) Is \( P \lor (Q \to (P \land Q)) \) a tautology? Explain your answer.

3. Consider the implication \( A \to (B \lor C) \). Use De Morgan’s laws to simplify
   (a) the contrapositive
   (b) the converse

4. Negate the following proposition. Simplify as far as possible. Then determine which is true, the proposition or its negation.
   \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, [(x \leq y) \land (x + y = 0)] \)

5. Define the following terms.
   (a) the symmetric difference of two sets
   (b) logically equivalent
   (c) Boolean algebra
6. Give an example of a Boolean algebra whose associated set contains exactly four elements. Indicate the elements in the associated set and define the operations.

7. (a) Let $x$ and $y$ be any two elements in a Boolean algebra. Without skipping any steps, and using the format in the text, prove that

\[(x + y) \cdot (x \cdot y) = x \cdot \overline{x} + \overline{x} \cdot y.
\]

(b) Prove also that $(x \cdot y) + (x + y) = (x + \overline{y}) \cdot (x + y)$.

2.3 Projects

**Mathematics**

1. Produce a careful proof for the associative property for Boolean algebras. Use the hint given in Exercise 6 in Exercises 2.5.3 (page 67).

2. Write a report that explores the relationship between the logic operators AND, OR, and NOT and the set operations of intersection, union, and set complement.

3. Produce a Boolean algebra whose associated set is $\{0, a, b, c, d, e, f, g, h, i, j, k, l, m, n, 1\}$. Produce the addition and multiplication tables and also a table listing the complement of each element.

**Computer Science**

1. Write a report about the essential role the logic operators AND, OR, NOT, and the implication IF ... THEN ... play in computer programming. Give examples written in one or more computer languages.

2. Use your favorite computer language to write a program that accepts a proposition as input and outputs a truth table for the proposition.

3. Use your favorite object-oriented language to create a class whose instantiations represent sets. Provide methods for all the common set operations.

**General**

1. Examine newspapers, magazines, or broadcast interviews or newscasts. Look for at least five logical fallacies or logical errors. Write detailed analyses of each.

2. Write an essay about deductive and inductive reasoning. Define each form of reasoning, and then explain and illustrate the differences.

3. Find several real-life examples illustrating misuses of the logic operators AND and OR.

4. Discuss the ways in which the system of symbolic logic presented in this text serves as a model of the manner in which people think and reason. In what ways does it fail to reflect human thought and reason?

5. Children grow physically and mature socially over time. They also exhibit a process of mental development. Write an expository report describing the development of logical reasoning abilities in children.

2.4 Solutions to Sample Exam Questions

1. (a) $A \cap B = \{c, e\}$
   (b) $A - B = \{a, d\}$
   (c) $|A| = 4$
   (d) $\overline{A} = \{b, f, g\}$
   (e) $P(C) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

2. (a) $P \oplus Q \quad P \land Q \quad Q \rightarrow (P \land Q) \quad P \lor [Q \rightarrow (P \land Q)]$

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(b) The statement $P \lor [Q \rightarrow (P \land Q)]$ is not a tautology. The third row of the truth table does not have a T in the final column. That is, $P \lor [Q \rightarrow (P \land Q)]$ does not evaluate to true for every possible combination of truth values for $P$ and $Q$.

3. (a) The contrapositive of $A \rightarrow (B \lor C)$ is $\neg (B \lor C) \rightarrow \neg A$.
   It simplifies to $\neg B \land \neg C) \rightarrow \neg A$.

(b) The converse of $A \rightarrow (B \lor C)$ is $(B \lor C) \rightarrow A$.

4. $(\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, [(x \leq y) \land (x + y = 0)])$
   $\Rightarrow \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, [(x \leq y) \land (x + y = 0)]$
   $\Rightarrow \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \neg [(x \leq y) \land (x + y = 0)]$
   $\Rightarrow \exists x, y \in \mathbb{Z}, [(x \neq y) \lor (x + y = 0)]$

The negation is true. There are many examples to demonstrate this. One simple example is to consider $x = 1$. The only value of $y$ for which $x + y = 0$ is $y = -1$. However, $1 > -1$ and for every other value of $y, x + y \neq 0$. Thus, the negation is true for $x = 1$. (You could also say that $x = 1$ is a counterexample to the original proposition.)

5. (a) the symmetric difference of two sets: see page 27
   (b) logically equivalent: see page 47
   (c) Boolean algebra: see page 59

6. See Example 2.30 on page 61.
7. (a) The solution given here is not unique.

\[(x + y) \cdot (x \cdot y)\]  
\[= (x \cdot y) \cdot (x + y)\]  
commutativity  
\[= (x \cdot y) \cdot x + (x \cdot y) \cdot y\]  
distributivity  
\[= x \cdot (x \cdot y) + y \cdot (x \cdot y)\]  
commutativity (twice)  
\[= x \cdot (\overline{x} + \overline{y}) + y \cdot (\overline{x} + \overline{y})\]  
De Morgan (twice)  
\[= (x \cdot \overline{x} + x \cdot \overline{y}) + (y \cdot \overline{x} + y \cdot \overline{y})\]  
distributivity (twice)  
\[= (0 + x \cdot \overline{y}) + (y \cdot \overline{x} + 0)\]  
complement (twice)  
\[= (x \cdot \overline{y} + 0) + (y \cdot \overline{x} + 0)\]  
complement (twice)  
\[= x \cdot \overline{y} + y \cdot \overline{x}\]  
commutativity  
\[= x \cdot \overline{y} + \overline{x} \cdot y\]  
commutativity

(b) The Boolean expression

\[(x \cdot y) + (x + y) = (x + \overline{y}) \cdot (\overline{x} + y)\]

is the dual of the Boolean expression

\[(x + y) \cdot (x \cdot y) = x \cdot \overline{y} + \overline{x} \cdot y.\]

The duality principle guarantees that both will be true if either is true. Part (a) shows that

\[(x + y) \cdot (x \cdot y) = x \cdot \overline{y} + \overline{x} \cdot y\]

is true. Consequently,

\[(x \cdot y) + (x + y) = (x + \overline{y}) \cdot (\overline{x} + y)\]

is also true.
## 2.4.1 Fundamental Properties

### Fundamental Set Properties

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
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</thead>
<tbody>
<tr>
<td><strong>Idempotence</strong></td>
<td>$A \cup A = A$</td>
</tr>
<tr>
<td></td>
<td>$A \cap A = A$</td>
</tr>
<tr>
<td><strong>Domination</strong></td>
<td>$A \cup U = U$</td>
</tr>
<tr>
<td></td>
<td>$A \cap \emptyset = \emptyset$</td>
</tr>
<tr>
<td><strong>Associativity</strong></td>
<td>$(A \cup B) \cup C = A \cup (B \cup C)$</td>
</tr>
<tr>
<td></td>
<td>$(A \cap B) \cap C = A \cap (B \cap C)$</td>
</tr>
<tr>
<td><strong>Identity</strong></td>
<td>$A \cup \emptyset = A$</td>
</tr>
<tr>
<td></td>
<td>$A \cap U = A$</td>
</tr>
<tr>
<td><strong>Commutativity</strong></td>
<td>$A \cup B = B \cup A$</td>
</tr>
<tr>
<td></td>
<td>$A \cap B = B \cap A$</td>
</tr>
<tr>
<td><strong>De Morgan’s Laws</strong></td>
<td>$A \cup \overline{B} = \overline{A \cap B}$</td>
</tr>
<tr>
<td></td>
<td>$A \cap \overline{B} = \overline{A \cup B}$</td>
</tr>
<tr>
<td><strong>Distributivity (\cap over \cup)</strong></td>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
</tr>
<tr>
<td></td>
<td>$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$</td>
</tr>
<tr>
<td><strong>Complement</strong></td>
<td>$A \cup \overline{A} = U$</td>
</tr>
<tr>
<td></td>
<td>$A \cap \overline{A} = \emptyset$</td>
</tr>
<tr>
<td><strong>Involution</strong></td>
<td>$\overline{\overline{A}} = A$</td>
</tr>
</tbody>
</table>

### Fundamental Logical Equivalences

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Idempotence</strong></td>
<td>$(P \lor P) \Leftrightarrow P$</td>
</tr>
<tr>
<td></td>
<td>$(P \land P) \Leftrightarrow P$</td>
</tr>
<tr>
<td><strong>Associativity</strong></td>
<td>$[(P \lor Q) \lor R] \Leftrightarrow [(P \lor (Q \lor R))]$</td>
</tr>
<tr>
<td></td>
<td>$[(P \land Q) \land R] \Leftrightarrow [(P \land (Q \land R))]$</td>
</tr>
<tr>
<td><strong>Commutativity</strong></td>
<td>$(P \lor Q) \Leftrightarrow (Q \lor P)$</td>
</tr>
<tr>
<td></td>
<td>$(P \land Q) \Leftrightarrow (Q \land P)$</td>
</tr>
<tr>
<td><strong>Law of Simplification</strong></td>
<td>$[(P \land Q) \rightarrow P] \Leftrightarrow T$</td>
</tr>
<tr>
<td></td>
<td>$[(P \land Q) \rightarrow Q] \Leftrightarrow T$</td>
</tr>
<tr>
<td><strong>De Morgan’s Laws</strong></td>
<td>$\neg(P \lor Q) \Leftrightarrow [(\neg P) \land (\neg Q)]$</td>
</tr>
<tr>
<td></td>
<td>$\neg(P \land Q) \Leftrightarrow [(\neg P) \lor (\neg Q)]$</td>
</tr>
<tr>
<td><strong>Distributivity (\lor over \land)</strong></td>
<td>$[(P \land (Q \lor R))] \Leftrightarrow [(P \land Q) \lor (P \land R)]$</td>
</tr>
<tr>
<td></td>
<td>$[(P \lor (Q \land R)) \Leftrightarrow [(P \lor Q) \lor (P \lor R)]$</td>
</tr>
<tr>
<td><strong>Law of the Excluded Middle</strong></td>
<td>$[P \lor (\neg P)] \Leftrightarrow T$</td>
</tr>
<tr>
<td><strong>Law of Double Negation (Involution)</strong></td>
<td>$\neg(\neg P) \Leftrightarrow P$</td>
</tr>
<tr>
<td><strong>Law of Contradiction</strong></td>
<td>$[P \land (\neg P)] \Leftrightarrow F$</td>
</tr>
<tr>
<td><strong>Law of Addition</strong></td>
<td>$[P \rightarrow (P \lor Q)] \Leftrightarrow T$</td>
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</table>
DEFINITION 2.26 Boolean Algebra

A Boolean algebra, $\mathbb{B}$, consists of an associated set, $B$, together with three operators and four axioms. The binary operators $+$ and $\cdot$ map elements of $B \times B$ to elements of $B$. The unary operator complement, $\overline{\cdot}$, maps elements of $B$ to elements of $B$.

Parentheses have the highest precedence. The complement operator, $\overline{\cdot}$, has the second highest precedence, followed by $\cdot$ and then by $+$, the operator with lowest precedence.

The axioms are as follows:

Identity  There exist distinct elements, 0 and 1, in $B$ such that for every $x \in B$

\[
x + 0 = x \\
x \cdot 1 = x
\]

Complement  For every $x \in B$, there exists a unique element $\overline{x} \in B$ such that

\[
x + \overline{x} = 1 \\
x \cdot \overline{x} = 0
\]

Commutativity  For every pair of (not necessarily distinct) elements $x, y \in B$

\[
x + y = y + x \\
x \cdot y = y \cdot x
\]

Distributivity  For every three elements $x, y, z \in B$ (not necessarily distinct)

\[
x \cdot (y + z) = x \cdot y + x \cdot z \\
x + y \cdot z = (x + y) \cdot (x + z)
\]

The Duality Principle for Boolean Algebras

Let $T$ be a theorem that is valid over a Boolean algebra. Then if all 0s and 1s are exchanged, and if all $+$ and $\cdot$ are exchanged (with a suitable change in parentheses to preserve operator precedence), the result is a theorem that is also valid over the Boolean algebra.

Fundamental Boolean Algebra Properties

<table>
<thead>
<tr>
<th>Idempotence</th>
<th>Domination</th>
<th>De Morgan’s Laws</th>
<th>Absorption</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x + x = x$</td>
<td>$x + 1 = 1$</td>
<td>$\overline{x + y} = \overline{x} \cdot \overline{y}$</td>
<td>$x + x \cdot y = x$</td>
</tr>
<tr>
<td>$x \cdot x = x$</td>
<td>$x \cdot 0 = 0$</td>
<td>$\overline{x \cdot y} = \overline{x} + \overline{y}$</td>
<td>$x \cdot (x + y) = x$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Associativity</th>
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<tbody>
<tr>
<td>$(x + y) + z = x + (y + z)$</td>
<td>$x \cdot (y + z) = x \cdot (x + y)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(x \cdot y) \cdot z = x \cdot (y \cdot z)$</td>
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