ON COLORING CLAW-FREE GRAPHS

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Abstract. A graph G is *k-claw-free* if no vertex has k pairwise nonadjacent neighbors. A *proper k-coloring* of G is a map from the vertex set to a set of k *colors* such that no two adjacent vertices map to the same color. The *chromatic number* $\chi(G)$ is the minimal k such that a proper k-coloring of G exists. The *clique number* $\omega(G)$ is the maximum size of a set of vertices in G that are all pairwise adjacent. The *independence number* $\alpha(G)$ is the maximum size of a set of vertices in G that are all pairwise non-adjacent.

A theorem of Erdős implies that we can make $\chi(G)$ arbitrarily large while fixing $\omega(G) = 2$. Therefore in general we cannot bound $\chi(G)$ above by a function of $\omega(G)$. In 2005, Chudnovsky and Seymour showed that if G is 3-claw-free with $\alpha(G) \geq 3$, then $\chi(G) \leq 2\omega(G)$. We extend this result to k-claw-free graphs.

A 4-*hole* is four vertices joined together in a square, but neither of the diagonals is joined. We show that if G is k-claw-free, and 4-hole-free, with $\alpha(G) \geq k$, then

$$\chi(G) \leq \binom{k}{2}\,\omega(G) - 1.$$

We also exhibit a bound for the case where we admit 4-holes.

1. Introduction

Let G be a finite simple graph. The set of vertices and edges of G are denoted $V(G)$ and $E(G)$, respectively. For an integer k, a function $c : V(G) \to \{1, \ldots, k\}$ is a *k-coloring* of G. A *proper coloring* of G is a coloring such that no two adjacent vertices have the same color. The *chromatic number* $\chi(G)$ of a graph G is the minimal k such that a proper k-coloring of G exists. Similarly, a *proper k-edge-coloring* of G is a function $c : E(G) \to \{1, \ldots, k\}$ such that no two distinct edges that share a vertex have the same color. The *girth* of a graph is the length of its shortest cycle.

The following is a well-known result due to Erdős in 1959.

Theorem 1. *Given any positive numbers g and k, there exists a graph with girth at least g and chromatic number at least k.*

A *clique* in G is a set of vertices that are all pairwise adjacent. The *clique number* $\omega(G)$ is the maximum size of a clique in G. If G is a graph with girth $g \geq 4$, then $\omega(G) \leq 2$. By Theorem 1, however, the chromatic number $\chi(G)$ can be taken to be arbitrarily large. Therefore, in general, we cannot bound $\chi(G)$ above by a function of $\omega(G)$. In light of this, it is interesting to establish any bound of this form for specific classes of graphs.

The family of line graphs is a class for which a bound of $\chi(G)$ by a function of $\omega(G)$ from above is known. The *line graph* $L(G)$ of a graph G is the graph with vertex set $V(L(G)) := E(G)$ and where $a, b \in E(G)$ are adjacent as vertices of $L(G)$ if and only if, as edges in G, a and b have a common incident vertex. The *degree* $d(v)$ of a vertex $v \in V(G)$ is the number of neighbors of v. The *maximum degree* $\Delta(G)$ is the maximum of the degrees of the vertices of G. Shannon [3] proved that the edges of any graph can be properly colored using no more than $\frac{3}{2}\Delta(G)$ colors. This implies that if G is a line graph, then

$$\chi(G) \leq \tfrac{3}{2}\omega(G).$$

This result is tight as witnessed by a pentagon C_5.

Another example for which a linear bound is known is the class of claw-free graphs. A vertex set is *stable* if the vertices are pairwise nonadjacent. The *independence number* $\alpha(G)$ is the maximum size of a stable set. We say that $X \subseteq V(G)$ is a k-claw if the subgraph of G induced on X is the complete bipartite graph $K_{1,k}$. A graph G is *k-claw-free* if no subset of $V(G)$ is a k-claw. We may refer to a 3-claw simply as a *claw*, and 3-claw-free as *claw-free*. Claw-free graphs are well-known generalizations of line graphs. Chudnovsky and Seymour [1] in 2005 proved the following.

Theorem 2. *Let G be a connected claw-free graph with $\alpha(G) \geq 3$. Then*

$$\chi(G) \leq 2\omega(G),$$

and this is asymptotically best possible.

2. On 4-Hole-Free, k-Claw-Free Graphs

We attempt to generalize this linear bound to k-claw-free graphs. However, we need to require an additional constraint. We say that $X \subseteq V(G)$ is a 4-*hole* if the subgraph of G induced on X is a 4-cycle. A graph G is 4-*hole-free* if no subset of $V(G)$ is a 4-hole. For $v \in V(G)$, we denote the set of neighbors of v in G by $N_G(v)$.

Theorem 3. *Let G be k-claw-free, 4-hole free, and connected, with $\alpha(G) \geq k \geq 3$. Then $\Delta(G) \leq \binom{k}{2}\omega(G) - 1$, and consequently $\chi(G) \leq \binom{k}{2}\omega(G) - 1$.*

Proof. Brook's Theorem states that $\chi(G) \leq \Delta(G)$ as long as G is not complete nor an odd cycle. It is clear that G cannot be complete, and that for an odd cycle, the second result hold. Therefore the second statement follows from the first statement by Brook's Theorem.

To prove the first statement, we perform induction on $|V(G)|$. Let v be a vertex of degree $\Delta(G)$. Suppose all vertices besides v are joined to v. Then since $\alpha(G) \geq k$, there are k neighbors of v that are pairwise nonadjacent. Then we have a k-claw, a contradiction. Therefore there is $u \in V$, $u \neq v$, such that u is nonadjacent to v. We may choose u such that $G \backslash u$ is connected (by choosing one that has maximum distance from v, say). If $\alpha(G\backslash u) \geq k$, then the induction hypothesis gives

$$\Delta(G\backslash u) \leq \binom{k}{2}\omega(G\backslash u) - 1.$$

Since u and v are nonadjacent, $\Delta(G) = \Delta(G\backslash u)$, and also $\omega(G\backslash u) \leq \omega(G)$, hence the desired result follows.

Now we may assume that $\alpha(G\backslash u) = k-1$. Let $A = \{a_1, \ldots, a_{k-1}\} \subseteq V(G\backslash u)$ be stable. Let $N_i = N_{G\backslash u}(a_i)$ for $i = 1, 2, \ldots, k-1$. Let $M_i = N_i \backslash (\bigcup_{j \neq i} N_j)$ for $i = 1, 2, \ldots, k-1$, and let $M_{ij} = N_i \cap N_j$ for $1 \leq i, j \leq k-1$, $i \neq j$. Since $\alpha(G\backslash u) = k-1$, all vertices are either in A or are in N_i for some i. If $x, y \in M_i$ are nonadjacent, then $\{x, y\} \cup A\backslash a_i$ is a stable set of size k, a contradiction. Therefore $M_i \cup \{a_i\}$ is a clique. If $x, y \in M_{ij}$ is nonadjacent, then $\{x, y, a_i, a_j\}$ is a 4-hole, a contradiction. Therefore M_{ij} is a clique. Since the union of the M_i and the M_{ij} equals the union of the N_i, we have

$$V(G\backslash u) = A \cup \bigcup_{1 \leq i \leq k-1} M_i \cup \bigcup_{\substack{1 \leq i,j \leq k-1 \\ i \neq j}} M_{ij}.$$

Therefore

$$|V(G\backslash u)| \leq \sum_{1 \leq i \leq k-1} |M_i \cup \{a_i\}| + \sum_{\substack{1 \leq i,j \leq k-1 \\ i \neq j}} |M_{ij}|.$$

There are $k-1$ summands in the first sum and $\binom{k-1}{2}$ in the second. Each summand is bounded by $\omega(G\backslash u)$; therefore we get

$$|V(G\backslash u)| \leq \binom{k}{2}\omega(G\backslash u).$$

Recall that $\omega(G\backslash u) \leq \omega(G)$. Furthermore, $\Delta(G) \leq |V(G)| - 2$ because u is not adjacent to v. Therefore

$$\Delta(G) \leq |V(G\backslash u)| - 1 \leq \binom{k}{2}\omega(G) - 1,$$

as desired. □

3. On General k-Claw-Free Graphs

In the previous section, we generalized Theorem 2 to k-claw-free graphs by requiring the absence of 4-holes. In this section, we shall generalize Theorem 2 to k-claw-free graphs without adding this constraint. However, the bound we obtain here is not linear in the clique number.

We shall employ Ramsey theory in our proof. We denote a complete graph on n vertices as K_n.

Theorem 4 (Ramsey)**.** *There exists a minimal positive integer $R(p, q)$ with the following property. Let G be a complete graph on n vertices. Each edge is colored red or blue. If $n \geq R(p, q)$, then G contains a red K_p or a blue K_q. Moreover,*

$$R(p, q) \leq \binom{p + q - 2}{p - 1}.$$

Ramsey's Theorem is very well known; the bound for $R(p, q)$ can be found, for example, in Theorem 3.4 of [2].

Lemma 5. *Let G be a graph. If $\alpha(G) < k$, then*

$$|V(G)| < \binom{\omega(G) + k - 1}{k - 1}.$$

Proof. Let H be a complete graph on the vertex set $V(H) = V(G)$. Color an edge of H red if it is an edge in G, and blue otherwise. By Theorem 4, if $|V(H)| \geq R(\omega(G)+1, \alpha(G)+1)$, then H either has a red $K_{\omega(G)+1}$ or a blue $K_{\alpha(G)+1}$. So G either has a clique of size $\omega(G) + 1$ or a stable set of size $\alpha(G) + 1$, a contradiction. Therefore $|V(G)| = |V(H)| < R(\omega(G) + 1, \alpha(G) + 1) \leq \binom{\omega(G)+\alpha(G)}{\alpha(G)} \leq \binom{\omega(G)+k-1}{k-1}$. □

Theorem 6. *Let G be k-claw-free. Then $\chi(G) \leq \binom{\omega(G)+k-1}{k-1}$.*

Proof. If $k = 1$, then we have isolated vertices. If $k = 2$, then we have a disjoint union of complete graphs. Both of these cases are trivial to check.

Now assume $k \geq 3$. We will prove a slightly stronger result, that $\chi(G) \leq \binom{\omega(G)+k-1}{k-1} - 2$. Without loss of generality, we may assume that G is connected.

Brook's Theorem states that $\chi(G) \leq \Delta(G)$ as long as G is not complete nor an odd cycle. It is easy to check that if G is complete or an odd cycle, the desired result hold. Therefore it remains to show that $\Delta(G) \leq \binom{\omega(G)+k-1}{k-1} - 2$.

To prove this, we consider two cases. If $\alpha(G) < k$, then by Lemma 5,

$$\Delta(G) \leq |V(G)| - 1 \leq \binom{\omega(G)+k-1}{k-1} - 2,$$

as desired.

We may assume $\alpha(G) \geq k$. We proceed by induction on $|V(G)|$. Let v be a vertex of degree $\Delta(G)$. Suppose all vertices besides v are joined to v. Then since $\alpha(G) \geq k$, there are k neighbors of v that are pairwise nonadjacent. Then we have a k-claw, a contradiction. Therefore there is $u \in V$, $u \neq v$, such that u is nonadjacent to v. We may choose u such that $G\backslash u$ is connected (by choosing one that has maximum distance from v, say). If $\alpha(G\backslash u) < k$, then by the first case we have

$$\Delta(G\backslash u) \leq \binom{\omega(G\backslash u)+k-1}{k-1} - 2.$$

If $\alpha(G\backslash u) \geq k$, the induction hypothesis gives the same result. Since u and v are nonadjacent, $\Delta(G) = \Delta(G\backslash u)$, and also $\omega(G\backslash u) \leq \omega(G)$, the desired result follows. □

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