Vertex-Pancyclicity of Hypertournaments

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Received June 28, 2007; Revised April 9, 2009

Published online 2009 in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/jgt.20432

Abstract: A hypertournament or a k-tournament, on n vertices, $2 \leq k \leq n$, is a pair $T=(V, E)$, where the vertex set V is a set of size n and the edge set E is the collection of all possible subsets of size k of V, called the edges, each taken in one of its $k!$ possible permutations. A k-tournament is pancyclic if there exists (directed) cycles of all possible lengths; it is vertex-pancyclic if moreover the cycles can be found through any vertex. A k-tournament is strong if there is a path from u to v for each pair of distinct vertices u and v. A question posed by Gutin and Yeo about the characterization of pancyclic and vertex-pancyclic hypertournaments is examined in this article. We extend Moon's Theorem for tournaments to hypertournaments. We prove that if $k \geq 8$ and $n \geq k+3$, then a k-tournament on n vertices is vertex-pancyclic if and only if it is strong. Similar results hold for other values of k. We also show that when $n \geq 7$, $k \geq 4$, and $n \geq k+2$, a strong k-tournament on n vertices is pancyclic if and only if it is strong. The bound $n \geq k+2$ is tight. We also find bounds for the generalized problem when we extend vertex-pancyclicity to require d edge-disjoint cycles of each

Contract grant sponsors: Richter Memorial Funds (to J. Y.); California Institute of Technology (to J. Y.).

possible length and extend strong connectivity to require d edge-disjoint paths between each pair of vertices. Our results include and extend those of Petrovic and Thomassen.

Keywords: *hypertournaments; Hamiltonian cycles*

1. INTRODUCTION

Redei's Theorem and Camion's Theorem are two of the most well-known and important theorems regarding tournaments and Hamiltonicity.

Theorem 1 (Redei). *Every tournament has a Hamiltonian path.*

A tournament is *d-edge-connected* if, for any two distinct vertices $u,\ v \in V$, there are d pairwise edge-disjoint paths from u to v. It is *strongly connected*, or simply, *strong*, if it is 1-edge-connected.

Theorem 2 (Camion). *Every strong tournament has a Hamiltonian cycle.*

A tournament T is *pancyclic* if there exist cycles of all possible lengths; it is *vertex-pancyclic* if there exist cycles of all possible lengths containing each vertex of T. Moon's Theorem generalizes Camion's Theorem.

Theorem 3 (Moon). *Every strong tournament is vertex-pancyclic.*

The proof of Redei's Theorem is very simple, and a proof of Moon's Theorem can be found in [1]. A digraph is *semicomplete* if every pair of vertices has one or two edges between them. All three theorems above are valid for semicomplete digraphs.

Let V be a n-set. Let E be the collection of all possible k-subsets of V, $2 \leq k \leq n$, each taken in one of its $k!$ possible permutations. A pair $T=(V,E)$ is called a *hypertournament* or a *k-tournament*. Each element of V is a *vertex*, and each ordered k-tuple of E is a *hyperedge* or, simply, an *edge*.

For vertices $u, v \in V$ and an edge $e=(x_1, \ldots, x_k) \in E$, we say u *dominates* v via edge e if u precedes v in e, that is if $u=x_i$, $v=x_j$, $1 \leq i < j \leq k$. We denote this by uev. A *path* consists of an alternating sequence

$$x_0 e_1 x_1 e_2 x_2 \ldots x_{\ell-1} e_\ell x_\ell$$

of distinct vertices x_i and distinct edges e_i so that x_{i-1} dominates x_i via e_i, $i=1,\ldots,\ell$. Such a path has *length* ℓ. A *cycle* is a path when all vertices are distinct except $x_0=x_\ell$; the cycle has the same length ℓ. A path (cycle) of T is *Hamiltonian* if it contains all vertices of T. Let $V(X)$ and $E(X)$ denote the set of vertices and edges of X, respectively, where X could be a (hyper)tournament, path, or cycle.

It is natural for one to ask whether these theorems hold for hypertournaments as well. Gutin and Yeo [2] proved the following. The proof can also be found in Chapter 11 of [1].

Theorem 4. *Let $k \geq 3$.*

(i) *Every k-tournament on $n \geq k+1$ vertices has a Hamiltonian path.*
(ii) *Every strong k-tournament on $n \geq k+2$ vertices has a Hamiltonian cycle.*

Recently, Petrovic and Thomassen [3] proved the following generalization of (ii).

Theorem 5. *Let T be a d-edge-connected k-tournament on n vertices. If $n \geq k+1+24d$ for $k \geq 4$, and $n \geq 30d+2$ for $k=3$, then T has d edge-disjoint Hamiltonian cycles.*

Gutin and Yeo [2] mentioned as unsolved the problem of deciding if a k-tournament is pancyclic or vertex-pancyclic. Specifically, we want to find all pairs (n,k) such that every strong k-tournament of order n is pancyclic (vertex-pancyclic, resp.). Petrovic and Thomassen [3] characterized the vertex-pancyclic k-tournaments.

Theorem 6. *If $k \geq 4$ and $n \geq k+25$ or if $k=3$ and $n \geq 32$, then T is vertex-pancyclic if and only if T is strong.*

However, Petrovic and Thomassen's characterization of vertex-pancyclic k-tournaments is incomplete in that it is only proved for sufficiently large n. In this article, we will improve the bound that was given. Furthermore, regarding the pancyclicity question posed by Gutin and Yeo, we will give a necessary and sufficient condition for a k-tournament to be pancyclic. However, our characterization is only for $n \geq 7$ and $k \geq 4$. Finally, we will improve upon the generalization made by Petrovic and Thomassen in Theorem 5.

2. ON VERTEX-PANCYCLICITY

Petrovic and Thomassen answered the primary problem of deciding if a k-tournament is vertex-pancyclic for $n \geq k+25$, $k \geq 4$ and for $n \geq 32$, $k=3$; we relax these restrictions of n. Their proof applied Hall's Theorem first to provide disjointness of hyperedges, then remove hyperedges used by the path P defined in Theorem 9 below. We reverse the order, applying Hall's Theorem after removing the hyperedges of P.

Lemma 7. *Let T be a 3-tournament and P a path of T. A pair of distinct vertices x and y can be in at most four of the hyperedges of P.*

Proof. Let $P = v_0 e_1 v_1 \ldots e_\ell v_\ell$. Suppose $x = v_m$ and $y = v_n$ (remember that vertices are distinct). As the hyperedges of a 3-tournament are triplets, if a hyperedge e_i of P contains both x and y, at least one of the vertices is an endpoint of the hyperedge in P. Therefore the hyperedges of P containing both x and y are all incident upon v_m or v_n in P. Hence there are at most four hyperedges of a path that contains a specific pair of distinct vertices.

If one of x and y is not a vertex of P (respectively, neither of them are), then it is clear that the maximum number of hyperedges containing both x and y is reduced to two (respectively, zero). ■

Lemma 8. *If*

(i) $k \geq 8$ *and* $n \geq k+3$,
(ii) $k \geq 5$ *and* $n \geq k+4$,
(iii) $k=4$ *and* $n \geq 11$, *or*
(iv) $k=3$ *and* $n \geq 15$,

then

$$\binom{k}{2} \leq \left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil - (n-1) \quad \textit{for } k \geq 4 \tag{*}$$

and

$$\binom{k}{2} \leq \left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil - 4 \quad \textit{for } k=3$$

Proof. It is trivial to check the case $k=3$. Assume $k \geq 4$. If $n=k+3$, (*) can be rearranged to give

$$k^2+k+3=2\left(\binom{k}{2}+k+2\right)-1 \leq \binom{n-2}{k-2}=\binom{k+1}{k-2}=\frac{1}{6}(k^3-k)$$

It is easy to verify that $k^3-6k^2-7k-18 \geq 0$ for $k \geq 8$. Similarly, for $n=k+4$, $k \geq 5$ suffices. For $k=4$, (*) holds with $n=11$. Now we apply induction on n; we wish $\binom{n-2}{k-2}-2\binom{k}{2}-2n+3 \geq 0$. We have established the respective base cases above. Increasing n by 1 increases $\binom{n-2}{k-2}-2\binom{k}{2}-2n+3$ by $\binom{n-1}{k-2}-\binom{n-2}{k-2}-2=\binom{n-2}{k-3}-2$. This is nonnegative for $k \geq 4$, $n \geq k+1$. ■

Remark. The ceiling is actually required for the case of $k=5$, $n=9$; the inequality does not hold when the ceiling is removed.

We form an ordinary tournament (2-tournament) M from T with vertex set $V(T)$ in the following way. For $u, v \in V(T)$, orient the edge uv from u to v if u dominates v in at least half of the hyperedges of T containing u and v. M is called the *majority digraph* of T. In the case where u dominates v in exactly half of the hyperedges, M will have an edge from u to v and another edge from v to u. Therefore, it is possible that the majority digraph is not strictly a tournament. However, this distinction is immaterial to the following proofs. Alternatively, one can randomly choose one of the two edges to exclude from M to guarantee that the majority digraph is a tournament. This notion was first introduced in [2].

Theorem 9. *Let T be a k-tournament on n vertices. If*

(i) $k \geq 8$ *and* $n \geq k+3$,
(ii) $k \geq 5$ *and* $n \geq k+4$,
(iii) $k=4$ *and* $n \geq 11$, *or*
(iv) $k=3$ *and* $n \geq 15$,

then T is vertex-pancyclic if and only if T is strong.

Proof. It is obvious from the definition that a vertex-pancyclic T is strong. Now assume that T is strong. Fix a vertex x of T and a length $\ell \in \{3,4,\ldots,n\}$; we shall find an ℓ-cycle of T through x. By construction of M, if u dominates v in M, then u dominates v via $\left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil$ hyperedges of T. Call these the *corresponding hyperedges*.

Assume first that $k>3$. If M is strong, then by Moon's Theorem M has an ℓ-cycle C' of M through x. Pick a corresponding hyperedge for each edge of C'. Lemma 8 guarantees that we may pick them all distinct, and then we have an ℓ-cycle C in T. Thus we may assume that M is not strong.

The relation that two vertices u and v are strongly connected (that there exists a $u \to v$ path and a $v \to u$ path) is an equivalence relation; call the equivalence classes the *strong components*. Let $S_1,\ldots,S_t$ be the strong components of M. It is well known that we can order these components such that there are no edges from S_j to S_i, $1 \le i < j \le t$. We say these strong components are *canonically ordered*. S_1 and S_t are called the *initial* and *terminal strong components* of M, respectively.

Because T is strong, there exists a path $P = x_0e_1x_1e_2x_2\ldots e_px_p$ in T connecting a vertex from the terminal component S_t to the initial component S_1. Adding the p edges $\{x_0x_1, x_1x_2, \ldots, x_{p-1}x_p\}$ to M, we obtain a strong semicomplete digraph D. As Moon's Theorem (a strong tournament is vertex-pancyclic) extends to strong semicomplete digraphs (see [1]), there exists an ℓ-cycle C' of D through x. We will form a cycle C of T from C' by using the same vertex set (in the same permutation). The only condition we need to check is that no edges are repeated. For an edge $x_{i-1}x_i$ of C' that originated from P, we use e_i for C; note that these are distinct. For the remaining edges of C', recall that each one has $\left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil$ corresponding hyperedges. We form a bipartite graph G with partite classes A and B. Every pair of vertices in T is a vertex in A. Every k-subset of vertices in T is a vertex in B. A vertex in A is joined to a vertex in B if the corresponding pair of vertices is contained in the corresponding k-subset. Since P has at most $n-1$ hyperedges, after removing the hyperedges in P from B in the bipartite graph G, the vertices in A have degree at least $\left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil - (n-1)$. The vertices in B have degree $\binom{k}{2}$; thus by Hall's marriage Theorem, if $\binom{k}{2} \le \left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil - (n-1)$, then there is a complete matching from A into B. By Lemma 8, the inequality is satisfied. Thus we can choose a hyperedge for each remaining edge of C' such that all the hyperedges chosen (including those from P) are distinct. Therefore by using these hyperedges, we have an ℓ-cycle C of T through x. As ℓ and x are arbitrary, T is vertex-pancyclic.

Assume now $k=3$. We repeat the first part of the proof. Again, P may have as many as $n-1$ hyperedges. However, by Lemma 7, at most 4 of the corresponding hyperedges for each pair of vertices are in P. Therefore we require $\binom{k}{2} \le \left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil - 4$ instead, which is satisfied by Lemma 8. ■

Remark. Gutin and Yeo [2] proved that for $k \ge 3$, a hypertournament with $n \ge k+1$ has a Hamiltonian path and strong one with $n \ge k+2$ has a Hamiltonian cycle; see Theorem 4 above. These bounds are tight: It is obvious that there are no Hamiltonian paths for $n=k$; Gutin and Yeo constructed in [2] a strong k-tournament with $n=k+1$ vertices yet admits no Hamiltonian cycle.

For $k \geq 8$, this theorem says a strong k-tournament with $n \geq k+3$ vertices is vertex-pancyclic. We also know that this statement is false for $n = k+1$. For the $n = k+2$ case, it is currently unknown whether the statement holds. We will, however, fill this gap with pancyclicity in the following section. Also, for $k < 8$, there are a few more cases that have not been decided.

3. ON PANCYCLICITY

A k-tournament is *pancyclic* if it contains cycles of all possible lengths. For sufficiently large n and k, a strong k-tournament T with $n = k+2$ vertices is pancyclic. To show this, we will modify Lemmas 3.2 and 3.4 of [2], which were used to prove the second part of Theorem 4.

Given a strong k-tournament T and its majority digraph M, we say that (P, Q) is an *ℓ-cyclic pair of paths* if P is an $x \to y$ path in T and Q is a $y \to x$ path in M such that $V(P) \cap V(Q) = \{x, y\}$ and $|V(P) \cup V(Q)| = \ell$. We also require that x and y be in different strong components of M if M is not strong.

For a path P and $u, v \in V(P)$, uPv is the segment of P from u to v. For paths P and Q such that the terminal vertex of P and the initial vertex of Q are the same or are separated by a single edge (as in the case of a 2-tournament), PQ is the path by adjoining Q after P.

Lemma 10. *Let $k \geq 3$ and $3 \leq \ell \leq n$. For every strong k-tournament with n vertices, there exists an ℓ-cyclic pair of paths.*

Proof. Let T be a strong k-tournament with n vertices. Suppose the majority digraph M of T is strong. By Moon's Theorem, M is vertex-pancyclic; hence, there exists an ℓ-cycle $R = x_1x_2 \ldots x_\ell x_1$ in M. Then $P = x_1ex_2$, where e is an $x_1 \to x_2$ edge, and $Q = x_2Rx_1$ form an ℓ-cyclic pair of paths.

Now we assume that M is not strong, and let $S_1, \ldots, S_t$ be the canonically ordered strong components of M (as in Theorem 9). Let $R = x_1e_1x_2e_2 \ldots e_{m-1}x_m$ be the shortest $S_t \to S_1$ path in T. Then $x_1 \in S_t$, $x_m \in S_1$, and $\{x_2, \ldots, x_{m-1}\} \cap (S_1 \cup S_t) = \emptyset$. If $m \geq \ell$, then x_ℓ is not in S_t, and that there are no edges from S_t, which contains x_1, to the strong component containing x_ℓ. Therefore, since M is semicomplete, $x_\ell x_1$ is an edge in M. Then $P = x_1Rx_\ell$ and $Q = x_\ell x_1$ form an ℓ-cyclic pair of paths, where the endpoints are in different strong components.

Now assume $m < \ell$, let $M' = M - \{x_2, x_3, \ldots, x_{m-1}\}$ and let $S'_1, \ldots, S'_{t'}$, $t' \leq t$, be the canonically ordered strong components of M'. Notice that M' is also semicomplete, and that $x_1 \in S'_{t'}$ and $x_m \in S'_1$ are still in the (new) terminal and initial components, respectively. We want a path Q of length $\ell - m + 1 \geq 2$ from x_m to x_1 in M'; then (R, Q) is an ℓ-cyclic pair of paths.

It remains to construct the desired path Q. Let j be minimum such that $s = |S'_1| + |S'_2| + \cdots + |S'_j| + |S'_{j+1}| \geq \ell - m + 1$. For each strong component S'_i, $1 \leq i \leq j$, we choose a Hamiltonian path Q'_i, whose existence is guaranteed by Camion's Theorem. For Q'_1, we stipulate that the path starts at $x_m \in S'_1$. Now we construct a path Q'_{j+1} such that $Q = x_mQ'_1Q'_2 \ldots Q'_jQ'_{j+1}x_1$ is of length $\ell - m + 1$. Notice that there is precisely one edge from the terminal vertex of Q'_i to the initial vertex of Q'_{i+1}, so the path Q is well-defined. If $j+1=t'$, then $x_1 \in S'_{j+1}$ and S'_{j+1} is strong, hence vertex-pancyclic (Moon's Theorem). Thus we can construct a path Q'_{j+1} in S'_{j+1} with $\ell-m+2-s+|S'_{j+1}|$ vertices, ending at x_1. Otherwise, if $j+1<t'$, let Q'_{j+1} be a path in S'_{j+1} with $\ell-m+1-s+|S'_{j+1}|$ vertices. Then indeed $|Q'_1|+|Q'_2|+\cdots+|Q'_j|+|Q'_{j+1}\cup\{x_1\}|=\ell-m+2$, so Q has length $\ell-m+1$, as desired. ■

For distinct vertices $u, v \in V(T)$, let $E_T(u,v)$ denote the set of edges of $E(T)$ in which u dominates v; the subscript T is omitted when the tournament is clear from context.

Theorem 11. *Every strong k-tournament with n vertices, where* $n \geq k+2 \geq 6$ *and* $n \geq 7$, *is pancyclic.*

Proof. For $n \geq k+2 \geq 6$, we have $\binom{n-2}{k-2} \geq 2n-4$ if and only if $n \geq 7$. Let T be a k-tournament with n vertices such that $n \geq k+2 \geq 6$ and $\binom{n-2}{k-2} \geq 2n-4$, and let M be the majority digraph of T.

Gutin and Yeo proved that T is Hamiltonian (Theorem 4). Thus we fix a length $\ell \in \{3,4,\ldots,n-1\}$, and show that there exists an ℓ-cycle in T.

We first suppose that M is strong. By Moon's Thereom, there exists an ℓ-cycle $C'=x_1x_2\ldots x_\ell x_1$ in M. For $i=1,2,\ldots,\ell$, we have $|E(x_{i-1},x_i)| \geq \frac{1}{2}\binom{n-2}{k-2} \geq n-2$, where we define $x_0=x_\ell$. If $\ell \leq n-2$, we can choose corresponding hyperedges e_j from T so that $C=x_1e_1x_2e_2\ldots e_{\ell-1}x_\ell e_\ell x_1$ is an ℓ-cycle in T. So we may assume that $\ell=n-1$. There exist distinct edges e_1 and e_2, such that $\{e_1,e_2\} \subseteq E(x_1,x_2)$. Also, since $k \leq n-2=\ell-1$, and since e_1 and e_2 do not contain the same set of vertices, one of e_1, e_2 does not contain a vertex in the set $\{x_3,x_4,\ldots,x_{\ell-1}\}$. Without loss of generality, assume $x_i \notin e_1$, where $i \in \{3,4,\ldots,\ell-1\}$. Since $|E(x_{i-1},x_i)| \geq \ell-1$, we can choose corresponding hyperedges f_j from T so that

$$P=x_i f_1 x_{i+1} f_2 \ldots x_\ell f_{\ell-i+1} x_1 e_1 x_2 f_{\ell-i+2} x_3 \ldots f_{\ell-2} x_{i-1}$$

is a path of length $\ell-1$. Since $x_i \notin e_1$, we have $e_1 \notin E(x_{i-1},x_i)$; since $|E(x_{i-1},x_i)| \geq \ell-1$, there is an edge $f_{\ell-1} \in E(x_{i-1},x_i)-E(P)$. Hence $C=Px_{i-1}f_{\ell-1}x_i$ is a cycle of length ℓ, as desired.

Now suppose that M is not strong. By Lemma 10, there exists an ℓ-cyclic pair of paths (P,Q), where $P=x_1e_1x_2e_2\ldots e_{p-1}x_p$ is a path in T and $Q=y_1y_2\ldots y_q$ is a path in M. Recall that $y_1=x_p$, $y_q=x_1$, and y_1, y_q are in different strong components of M. Fix i such that y_i and y_{i+1} are in different strong components of M. By definition of M, we have $|E(y_{j-1},y_j)| \geq \frac{1}{2}\binom{n-2}{k-2} \geq n-2 \geq \ell-1$ for $j>1$. Also, if $|E(y_i,y_{i+1})|=n-2$, then $|E(y_{i+1},y_i)| \geq n-2$; but y_i and y_{i+1} are in different strong components, thus $|E(y_i,y_{i+1})| \geq n-1 \geq \ell$. As $|E(y_{j-1},y_j)| \geq \ell-1$ for $j>1$, we can extend the path P to a path $R=r_1f_1r_2f_2\ldots f_{\ell-1}r_\ell$ in T with $r_1=y_{i+1}$, $r_2=y_{i+2},\ldots,$ $r_{q-i}=y_q=x_1$, $r_{q-i+1}=x_2,\ldots,$ $r_{\ell+1-i}=x_p=y_1$, $r_{\ell+2-i}=y_2,\ldots,$ $r_\ell=y_i$, using edges of P and choosing the remaining edges. Now as $|E(y_i,y_{i+1})| \geq \ell$, there is an edge $f_\ell \in E(y_i,y_{i+1})-E(R)$. Hence $Ry_if_\ell y_{i+1}$ is a cycle of length ℓ in T. ■

Remark. There is a small gap between the inequalities in this result and that of Theorem 4. Namely, for the cases of $k=3$ and $n<7$, a strong k-tournament on $n \geq k+2$ vertices is Hamiltonian, but it is not shown here whether it is pancyclic.

4. ON *d*-DISJOINT-VERTEX-PANCYCLICITY

A k-tournament is *d-disjoint-vertex-pancyclic* if each vertex of T is contained in d edge-disjoint ℓ-cycles for each possible length ℓ. Now we consider the generalized problem by extending vertex-pancyclicity to require d edge-disjoint cycles of each possible length and extending strong connectivity to require d edge-disjoint paths between each pair of vertices.

To prove the following lemma, we use the calculus of finite differences: The *finite forward difference* $\Delta_x f(x)$ of a function $f(x)$ with respect to x is defined as $\Delta_x f(x) = f(x+1) - f(x)$. The *n-th finite difference* is defined inductively as $\Delta_x^n f(x) = \Delta_x(\Delta_x^{n-1} f(x))$. If the function has more variables, they are held constant. This is the discrete analog of the derivative.

Lemma 12. *If $n \geq k+2d+1$ for $k \geq 8$, then*

$$d\left[\binom{k}{2}+(n-1)\right] \leq \left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil$$

Proof. Suppose $k \geq 8$ and $n \geq k+2d+1$, we want to show that

$$f = f(n,k,d) = \binom{n-2}{k-2} + 1 - 2d\left[\binom{k}{2}+n-1\right]$$

is nonnegative. Now

$$\begin{aligned}\Delta_n f(n,k,d) &= \binom{n-1}{k-2} + 1 - 2d\left[\binom{k}{2}+n\right] \\ &\quad - \left(\binom{n-2}{k-2} + 1 - 2d\left[\binom{k}{2}+n-1\right]\right) \\ &= \binom{n-2}{k-3} - 2d\end{aligned}$$

which is nonnegative when $n \geq k+2d+1$ and $k \geq 8$. Therefore it remains to show that f is nonnegative when $n = k+2d+1$.

To show that

$$g(k,d) = f(k+2d+1,k,d) = \binom{k+2d-1}{k-2} + 1 - 2d\left[\binom{k}{2}+k+2d\right]$$

is nonnegative, we shall show that $g(k,1)$, $\Delta_d g(k,1)$, $\Delta_d^2 g(k,1)$ and $\Delta_d^3 g(k,d)$ are all nonnegative for $k \geq 8$.

Now $g(k,1)=\binom{k+1}{k-2}+1-2\left[\binom{k}{2}+k+2\right]$ reduces to the case of $n=k+3$ considered in Lemma 8, and is nonnegative.

Note that

$$\binom{k+2d+1}{k-2}=\binom{k+2d+1}{2d+3}=\frac{k+2d+1}{2d+3}\frac{k+2d}{2d+2}\binom{k+2d-1}{2d+1}$$

hence

$$\begin{aligned}\Delta_d\, g(k,d) &= g(k,d+1)-g(k,d)\\ &=\binom{k+2d+1}{k-2}-2\left[\binom{k}{2}+k\right]-4(d+1)^2-\binom{k+2d-1}{2d+1}+4d^2\\ &=\left(\frac{k+2d+1}{2d+3}\frac{k+2d}{2d+2}-1\right)\binom{k+2d-1}{2d+1}-k^2-k-8d-4\end{aligned}$$

In particular,

$$\begin{aligned}\Delta_d g(k,1) &=\left(\frac{k+3}{5}\frac{k+2}{4}-1\right)\binom{k+1}{3}-k^2-k-8-4\\ &=\frac{1}{120}(k^5+5k^4-15k^3-125k^2-106k-1440)\end{aligned}$$

which has only one real root between 5 and 6, and is 37 when $k=6$. Similarly,

$$\begin{aligned}\Delta_d^2 g(k,d) &= \Delta_d\, g(k,d+1)-\Delta_d\, g(k,d)\\ &=\left[\left(\frac{k+2d+3}{2d+5}\frac{k+2d+2}{2d+4}-2\right)\left(\frac{k+2d+1}{2d+3}\frac{k+2d}{2d+2}\right)+1\right]\binom{k+2d-1}{2d+1}-8\end{aligned}$$

Then

$$\begin{aligned}\Delta_d^2\, g(k,1) &=\left[\left(\frac{k+5}{7}\frac{k+4}{6}-2\right)\left(\frac{k+3}{5}\frac{k+2}{4}\right)+1\right]\binom{k+1}{3}-8\\ &=\frac{1}{5040}(k^2+19k+76)(k+1)k(k-1)(k-2)(k-3)-8\end{aligned}$$

which is clearly increasing when $k\ge 4$, and is 20 when $k=5$. Finally,

$$\begin{aligned}\Delta_d^3 g(k,d) &= \Delta_d^2\, g(k,d+1)-\Delta_d^2\, g(k,d)\\ &=\frac{(k^2+8dk+17k+16d^2+56d+30)(k+4d+7)(k-2)(k-3)(k-4)}{(2d+7)(2d+6)(2d+5)(2d+4)(2d+3)(2d+2)}\\ &\quad\times\binom{k+2d-1}{2d+1}\end{aligned}$$

which is clearly nonnegative for $k\ge 4$, $d\ge 1$. Therefore $g(k,d)\ge 0$ for $k\ge 8$, and $f(n,k,d)$ is increasing in n for $n\ge k+2d+1$; we have that f is indeed nonnegative for $n\ge k+2d+1$, $k\ge 8$, as desired. ■

Theorem 13. *Let T be a k-tournament on n vertices. If*

(i) $k \geq 8$ *and* $n \geq k+2d+1$, *or*
(ii) $k=3$ *and* $n \geq 14d+1$,

then T is d-disjoint-vertex-pancyclic if and only if T is d-edge-connected.

Proof. We review the proof of Theorem 9. It is again obvious that a d-disjoint-vertex-pancyclic T is d-edge-connected. Now assume that T is d-edge-connected. We again consider $k>3$ first. Fix a vertex x of T and a length $\ell \in \{3,\ldots,n\}$; we shall find d disjoint ℓ-cycles of T through x. We again construct the majority digraph M. Previously, since T is strong, there exists a path P in T connecting a vertex x_1 from the terminal component S_t to a vertex x_2 in the initial component S_1. Now, because T is d-edge-connected, T has d edge-disjoint paths $P_1, P_2, \ldots, P_d$ connecting x_1 to x_2. We add the edges of P_i to M to obtain a strong semicomplete digraph D_i. Moon's Theorem gives us an ℓ-cycle C_i' through x. Now we choose corresponding hyperedges to get d disjoint ℓ-cycles of T through x. The edges arising only from the P_i can be directly used, as before. Since P_i has at most $n-1$ hyperedges, $P_1,\ldots,P_d$ can use at most $d(n-1)$ hyperedges. Since each remaining edge of M corresponds to $\left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil$ hyperedges, after removing the hyperedges in the P_i from B in the bipartite graph G, the vertices in A have degree at least $\left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil - d(n-1)$. Now we want to choose d corresponding hyperedges for each of the remaining edges of M such that all edges are disjoint. This can be accomplished by replicating A a total of d times. Let $A' = \{(a,i) : a \in A,\ 1 \leq i \leq d\}$, and join $(a,i) \in A'$ with $b \in B$ if and only if a and b are joined in G. Consider this new bipartite graph with partite classes A' and B. Now each vertex in B has degree $d\binom{k}{2}$. Since

$$d\binom{k}{2} \leq \left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil - d(n-1)$$

by Lemma 12, Hall's Marriage Theorem ensures that there exists a complete matching from A' to B. Thus for each edge e of M, we get d hyperedges of T by taking those matched to (e,i), $1 \leq i \leq d$. Now as these hyperedges are all distinct and disjoint from those of $P_1,\ldots,P_d$, we can use these to form edge-disjoint ℓ-cycles $C_1,\ldots,C_d$ through x in T.

For $k=3$, Lemma 7 allows us to replace $n-1$ with 4 in the above proof. Now $d\binom{k}{2} \leq \left\lceil \frac{1}{2}\binom{n-2}{k-2} \right\rceil - 4d$ becomes $3d \leq \left\lceil \frac{1}{2}(n-2) \right\rceil - 4d$, which is satisfied when $n \geq 14d+1$. ∎

ACKNOWLEDGMENTS

The author would like to thank Richard M. Wilson of the California Institute of Technology for his support in supervising this research. We also gratefully acknowledge the support by the Summer Undergrad Research Fellowships program of the same institute. This research is funded in part by the Richter Memorial Funds.

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