My main research interests are combinatorics and computational complexity. I study problems from various domains using combinatorial and computational tools. Since computational complexity is relevant in many contexts, I treasure the opportunities for interdisciplinary explorations and collaborations, especially in fields that I have little domain knowledge, so that I can continue to broaden my horizons by learning from my collaborators. Some of these projects are discussed in Section 3. On the other hand, purely theoretical research with no clear or immediate applications can also be valuable in pushing the boundaries of human understanding.

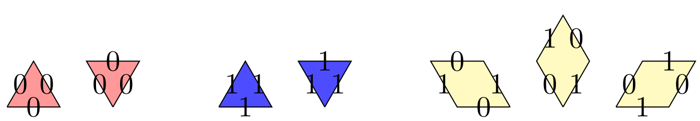
I study the computational complexity aspects of tilings in the context of discrete geometry and combinatorics, where tiles have simple geometries and the complexity arises from the combinatorial interactions of these simple tiles. Specifically, let Λ denote the cells of some lattice in Euclidean space. (We consider the triangular lattice in $\mathbb{R}^2$ and the cubic lattice in $\mathbb{R}^n$.) A **tile** is a finite subset of Λ. Let T be a finite set of tiles, called a **tileset**. A **tiling** π is a collection of translated[1] copies of tiles in T so that the tiles $\tau \in \pi$ are pairwise disjoint. A region $R \subseteq \Lambda$ is **tileable** (by T) if $R = \bigcup_{\tau\in\pi} \tau$ for some tiling π.

The main driving principle of my research program is that simple tiles can exhibit complex behaviors. I studied this phenomenon on the square lattice in the plane—which led to mentoring a related undergraduate research project—and the cubic lattice in higher dimensions; these are tersely summarized in Section 2 in the interest of brevity. In Section 1, I describe more recent work in which I studied tilings on the triangular lattice in the plane in several projects with five collaborators, three of whom are undergraduate students. Natural extensions for future projects are sprinkled throughout both sections, with a strong emphasis on problems suitable for undergraduate research.

I am especially interested in collaborating with undergraduate students in my research. Besides finding the mentoring relationship rewarding for me, I know that I can afford students the opportunity to further the frontiers of mathematics. A guiding principle for a wealth of appropriate problems is as follows. Each NP decision problem has a natural associated counting problem. For example, instead of asking for the existence of tilings, one could ask for the *number* of such tilings. A region is tileable if the number of tilings is positive. As such, being able to count in polynomial time implies that the decision problem is in P. Conversely, it seems intuitive that an NP-complete problem would have a #P-complete counting version.[2] However, it is unknown whether this is always true. As such, a good source of reasonable projects is to take an NP-complete problem and prove that its counting version is #P-complete.[3] This framework has led to a couple of successful summer research projects with undergraduates, as detailed in the following sections.

1. Triangular puzzle tilings

In this section, we tile on the triangular lattice in $\mathbb{R}^2$. Each cell is a unit equilateral triangle with three **edges**. Each edge on the boundary of tiles and regions is **labeled** by 0 or 1. Incident edges in a tiling must have equal labels.[4] Consider the seven **puzzle pieces**:



The number of tilings of triangular regions by puzzle pieces is a Littlewood–Richardson coefficient [KTW04]. These numbers are ubiquitous, naturally appearing in the study of symmetric

[1] If rotations or reflections are desired, we may simply add rotated or reflected tiles into T.

[2] #P is the class of counting problems associated to NP decision problems. See [Val79] for definitions and background.

[3] This is the expected result; to prove that it is not #P-complete would shock the world.

[4] These restrictions can easily be enforced geometrically by creating little notches to represent these labels.

functions, representation theory, and algebraic geometry. Since positivity of Littlewood–Richardson coefficients is in P [KT99], tileability of triangular regions by puzzle pieces is also in P. On the other hand, we showed that tiling general regions is more complicated:

Theorem 1 ([PY16]) *Tileability of general regions by puzzle pieces is* NP*-complete.*

Project 1 (Counting general tilings by puzzle pieces) A natural follow-up question is whether counting the number of tilings of general regions by puzzle pieces is #P-complete. This is a reasonable project at the undergraduate level by the guiding principle established in the introduction. Recently (summer 2017), together with Carleton Summer Research student Zephyr Lucas, we proved that the counting problem is #P-complete. This is achieved by tweaking the reduction in the NP-completeness proof to control the number of tilings in a predictable, though not 1–1, way. A much more ambitious follow-up is to consider tiling convex or simply connected regions.

Project 2 (Puzzle complexity) Littlewood–Richardson coefficients also appear as structure constants in the cohomology ring $H^*(\mathrm{Gr}_k(\mathbb{C}^n))$ of the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$. In fact, the puzzle pieces were introduced to understand Schubert calculus, the study of intersection theory on a Grassmannian. Other puzzle pieces were introduced in recent years to study equivariant cohomology and K-theory of Grassmannians [KT03, Knu10], Gromov–Witten invariants [BKT03], GL_n Belkale–Kumar coefficients [KP11], and equivariant cohomology of two-step flag varieties [Buc15]. Each of these tilesets can be analyzed for their computational complexity.

Project 3 (Tile discovery) Instead of studying tiles existing in the literature, we can also study new tilesets. We need a guiding principle for deciding which tiles to study. Since all the tilesets mentioned above determine structure constants for various rings, and structure constants obey certain *commutativity* and *associativity* relations, it is natural to seek **puzzle tilesets** that yield numbers that obey these rules in the hopes of discovering new rings (and even cohomology theories).

Recently (summer 2017), I led a group of three math and CS students (Zephyr Lucas, Anna Meyer, and Walt O'Connor) to consider this problem. The students devised and implemented algorithms to count puzzle tilings (which run much faster experimentally than naive brute force counting), created a program to generate random tilesets, and developed a distributed system to test commutativity and associativity relations to identify puzzle tilesets on as many computers as the college would allow us to use. While we did not discover any new puzzle tilesets by the end of summer, most of the known tilesets were successfully (re)discovered. Furthermore, since the students built the system so that the computational experiment can continue to run without human intervention, we can thus analyze the results at a later time after more data has been collected.

Natural follow-up directions include either improving the system (especially in the random generation subsystem to steer towards more likely tilesets) or proving that there are no other puzzle tilesets. Both of these directions are exciting (for conflicting reasons) and worth pursuing. The former is suitable for undergrads at any level while the latter is less accessible.

Project 4 (Understanding puzzles) Buch [Buc02] establishes combinatorial rules for calculating the product and coproduct of the K-theory of the Grassmannian using Young tableaux, a well-studied combinatorial object. Vakil gives a puzzle-theoretic rule for the product [Vak06]. Together with Pavlo Pylyavskyy, we prove a puzzle-theoretic rule for the coproduct [PY18]. We accomplish this by relating puzzles to Young tableaux. An ambitious multiyear project is to recast these proofs using *mosaics*, introduced by Purbhoo [Pur08], so the proofs are self-contained and completely based on tilings. The goal of this project is to give us a better understanding of the puzzles themselves, as opposed to only relying on the connection to other mathematical objects.

2. Square and cubic tilings

In this section, we consider tiling on the cubic lattice in $\mathbb{R}^n$. Some past work and future projects suitable for undergraduate research are briefly summarized.

Theorem 2 ([Yan13, §3]) *There is a tileset consisting of* 117 *rectangles*[5] *such that tileability of simply connected regions is* NP*-complete (and is* #P*-complete to count).*

Project 5 (Tiling simply connected regions by rectangles) Using techniques in combinatorial group theory developed by Conway and Lagarias [CL90] and extending the height function approach [Thu90], Rémila shows that tileability of simply connected regions is in P for any two fixed rectangles [Rém05] (see also [KK92]). It is natural to attempt to tighten the gap between 2 and 117 rectangles. Making incremental improvements to this end is well within reach of undergraduates, even if completely closing the gap seems to be beyond our current technology.

Project 6 (Counting tromino tilings) Tiling with straight trominoes is NP-complete [BNRR95]. Together with Kyle Meyer (University of Minnesota Research Experiences for Undergraduates (REU) student), we proved that the counting version is #P-complete.

A natural follow-up question is whether this remains true when tiling simply connected regions. This question is more difficult to answer, in part because tileability of simply connected regions by straight trominoes is in P. However, the project is not unreasonable as there are very simple tiling problems in P that are #P-complete to count. In particular, this can be seen in domino tilings in higher dimensions.

Project 7 (Dominoes in higher dimensions) Tiling by dominoes (two adjacent cells glued together) is equivalent to finding perfect matchings in the dual graph and so is in P in any dimension (see *e.g.* [LP09]). In the plane, tilings can be counted in polynomial time using a Pfaffian method [Kas67] (see also [Fis61, Kas61]). Using the fact that counting matchings in general graphs is hard [DL92], we proved the following:

Theorem 3 ([PY13a]) *Counting the number of tilings by a* $[2 \times 1 \times \cdots \times 1]$ *domino and its rotations is* #P*-complete in three dimensions and higher.*

We also proved that tileability by a $[2 \times \cdots \times 2 \times 1 \times \cdots \times 1]$ slab with rotations is NP-complete (and is #P-complete to count), answering a question asked by Moore [Moo]. Moreover, we proved that all these results remain true even when restricted to tiling contractible[6] regions. A natural extension is to generalize these results to tiling by an $[n_1 \times n_2 \times \cdots \times n_d]$ brick (with rotations) for any $n_1, \ldots, n_d \in \mathbb{N}$. This project is accessible to a motivated undergraduate, consisting of both low-hanging fruit and deeper questions.

3. Interdisciplinary projects

Preventing bacteria from developing resistance to antibiotics is of great importance in biology and medicine. One common method is to use an *antibiotic treatment plan* of a sequence of different antibiotics. A group of biologists and mathematicians formulated the creation of an effective treatment plan as an optimization problem, which they called the "antibiotics time machine" [MCG$^+$15]. Together with Ngoc Tran, we prove that creating an optimal antibiotic treatment plan is NP-hard [TY17], so we should not expect to find "fast" algorithmic solutions to this antibiotics problem. All is not lost: informed by such a result, the natural follow-up is to focus our efforts on finding heuristics that can solve this computational problem fast enough in practice to address real needs in medicine.

[5] In [PY13b], we establish the bound of 1000000 rectangles.

[6] Not only is a contractible region simply connected; all its homotopy groups are trivial.

Modern computer systems often have multiple processors. Often, the processors can be run in a variety of power states to balance the energy consumption with computational throughput. A *power management* policy may select the states on a per-core basis or as a whole. In the latter, scalability issues are becoming important as the number of cores in multiprocessor systems is increasing. Collaborating with electronics engineering researchers, we show that finding the optimal state combination given energy consumption constraints is NP-hard [PYJL15]. We also propose a heuristic algorithm based on combinatorial optimization. Experimental results show the heuristic beating the state-of-the-art algorithm using commercial benchmarks.

More recently, I worked with Qie He, an industrial and systems engineer, regarding computational challenges in controlling *Markov processes*. Viewing problems through the lens of computational complexity allows me to supervise and collaborate with undergraduate students with varied interests in adjoining fields. These problems are exciting as they often have practical, real-world applications that could impact our quality of life.

4. Conclusion

I have described several projects on tilings, including joint work with undergraduate students. We proved NP-completeness for tiling with puzzle pieces, with rectangles in simply connected regions, and with slabs in higher dimensions. We also showed #P-completeness for these and other problems. I have also proposed natural extensions, including NP-completeness or #P-completeness of tiling with d-dimensional bricks or other puzzle pieces, understanding puzzles via mosaics, and using tilings to discover new algebraic structures.

In addition to these compelling mathematical problems, what I find most enjoyable in my research is to work with students on combinatorial projects with a computational component. Furthermore, I would be excited to expand into any areas of particular interest to the faculty and students in your department and, more broadly, to leverage the liberal arts environment to initiate interdisciplinary collaborations.

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