

MATH 31A DISCUSSION

JED YANG

1. INTRODUCTION

Lecture 1

- Instructor: Calin Martin.
- Location: MOORE 100.

Sections 1E and 1F

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- Office: MS 6617A.
- Office hours: T 12:30–13:30 xor W 15:00–16:00.
- Discussion Location: DODD 78.
- Website: <http://www.math.ucla.edu/~jedyang/31a.1.09f/>.
- SMC: Sept. 30–Dec. 3, M–R 09:00–15:00, MS 3974, M 13:00–14:00.

2. ADMINISTRATION

- HW due Fridays in lecture, and I will hand back in section.
- Textbook: Rogawski, Single Variable Calculus, 2008.
- Determine office hour.

3. PRECALCULUS REVIEW

3.1. **Exercise 1.2.21.** Find the equation of the perpendicular bisector of the segment joining $(1, 2)$ and $(5, 4)$.

Solution. Slope of segment is $m_1 = \frac{4-2}{5-1} = \frac{1}{2}$. Slope of perpendicular bisector is $m_2 = -1/m_1 = -2$. Mid point is $(\frac{1+5}{2}, \frac{2+4}{2})$. So the equation can be written as $y - 3 = -2(x - 3)$. \square

3.2. **Exercise 1.2.23.** Find the equation of the line with x -intercept $x = 4$ and y -intercept $y = 3$.

Solution. Equation of the line is $y = mx + b$, where b is the y -intercept, hence $b = 3$. The x -intercept $x = 4$ will yield $y = 0$ (by definition), so substituting, we may solve for m . We get $0 = 4m + 3$, hence $m = -\frac{3}{4}$. So the equation can be written as $y = -\frac{3}{4}x + 3$. \square

3.3. **Exercise 1.2.24.** A line of slope $m = 2$ passes through $(1, 4)$. Find y such that $(3, y)$ lies on the line.

Solution. One way is to write down an equation of the line in point-slope form: $y = 2(x - 1) + 4$. Then we see clearly that if $x = 3$, then $y = 8$. Alternatively, the slope m is the change of y over the change of x . Symbolically, $m = \frac{\Delta y}{\Delta x}$, or $\Delta y = m\Delta x$. This concept will be useful later when we deal with differentials $dy = m dx$. Since the change in x is $\Delta x = 3 - 1 = 2$, we get that the change in y is $\Delta y = y - 4 = 2 \cdot 2 = 4$, hence $y = 8$. This method seems longer, but conceptually it is easier to do in one's head, and will lead to intuition for calculus later. \square

3.4. Exercise 1.4.55. Use the addition formulae for sine and cosine to prove

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \quad (1)$$

$$\cot(a - b) = \frac{\cot a \cot b + 1}{\cot b - \cot a}. \quad (2)$$

Proof. Recall that

$$\sin(a + b) = \sin a \cos b + \cos a \sin b \quad (3)$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b. \quad (4)$$

Now

$$\tan(a + b) = \frac{\sin(a + b)}{\cos(a + b)} \quad (5)$$

$$= \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b} \quad (6)$$

$$= \frac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a}{\cos a} \frac{\sin b}{\cos b}} \quad (7)$$

$$= \frac{\tan a + \tan b}{1 - \tan a \tan b}, \quad (8)$$

where we get from (6) to (7) by dividing top and bottom by $\cos a \cos b$.

The case for cotangent is completely analogous. Remember $\cot x = \frac{\cos x}{\sin x}$ and that $\sin(-b) = -\sin(b)$ and $\cos(-b) = \cos(b)$. Work out the details and convince yourself. \square

3.5. Exercise 1.4.56. Let θ be the angle between the line $y = mx + b$ and the x -axis. Prove that $m = \tan \theta$.

Proof. This is trivial. \square

3.6. Exercise 1.4.57. Let L_1 and L_2 be the lines of slope m_1 and m_2 , respectively. Show that the angle θ between L_1 and L_2 satisfies $\cot \theta = \frac{m_2 m_1 + 1}{m_2 - m_1}$.

Proof. This is immediate by using Exercises 55 and 56. \square

3.7. Exercise 1.4.58. Perpendicular Lines. Use Exercise 57 to prove that two lines with nonzero slopes m_1 and m_2 are perpendicular if and only if $m_2 = -1/m_1$.

Proof. What is $\cot(\pi/2)$? \square

3.8. **Exercise 1.4.59.** Apply the double-angle formula to prove:

$$(a) \cos \frac{\pi}{8} = \frac{1}{2} \sqrt{2 + \sqrt{2}}.$$

$$(b) \cos \frac{\pi}{16} = \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}}.$$

Guess the values of $\cos \frac{\pi}{32}$ and of $\cos \frac{\pi}{2^n}$ for all n .

Proof. Recall $\cos^2 t = \frac{1 + \cos(2t)}{2}$. For the general case, let $a_0 = 0$ and define inductively $a_n = \sqrt{2 + a_{n-1}}$. We claim that for $n \geq 1$, we have $\cos \frac{\pi}{2^n} = \frac{1}{2} a_{n-1}$. The base case is trivial. By induction, assume $\cos \frac{\pi}{2^n} = \frac{1}{2} a_{n-1}$. By the half-angle formula, we get $\cos \frac{\pi}{2^{n+1}} = \sqrt{\frac{1}{2} (1 + \frac{1}{2} a_{n-1})} = \sqrt{\frac{1}{4} (2 + a_{n-1})} = \frac{1}{2} a_n$. \square

4. BASIC LIMITS

4.1. **Basic Limit Laws.** Assume that $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ exist. Then:

(a) Sum Law:

$$\lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x).$$

(b) Constant Multiple Law: For any number $k \in \mathbb{R}$,

$$\lim_{x \rightarrow c} k f(x) = k \lim_{x \rightarrow c} f(x).$$

(c) Product Law:

$$\lim_{x \rightarrow c} (f(x)g(x)) = \left(\lim_{x \rightarrow c} f(x) \right) \left(\lim_{x \rightarrow c} g(x) \right).$$

(d) Quotient Law: If $\lim_{x \rightarrow c} g(x) \neq 0$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}.$$

4.2. **Exercise 2.3.22.** Evaluate the limit $\lim_{z \rightarrow 1} \frac{z^{-1} + z}{z + 1}$.

Solution. Recall that $\lim_{z \rightarrow 1} z = 1$ and $\lim_{z \rightarrow 1} 1 = 1$. By the Quotient Law, $\lim_{z \rightarrow 1} z^{-1} = \frac{\lim_{z \rightarrow 1} 1}{\lim_{z \rightarrow 1} z} = \frac{1}{1} = 1$. By the Sum Law, $\lim_{z \rightarrow 1} z^{-1} + z = \lim_{z \rightarrow 1} z^{-1} + \lim_{z \rightarrow 1} z = 1 + 1 = 2$. By the Sum Law, $\lim_{z \rightarrow 1} z + 1 = 2$. So by the Quotient Law, $\lim_{z \rightarrow 1} \frac{z^{-1} + z}{z + 1} = \frac{\lim_{z \rightarrow 1} z^{-1} + z}{\lim_{z \rightarrow 1} z + 1} = \frac{2}{2} = 1$. \square

4.3. **Exercise 2.3.29.** Can the Quotient Law be applied to evaluate $\lim_{x \rightarrow 0} \frac{\sin x}{x}$?

Solution. The Quotient Law requires the limit of the denominator, namely, $\lim_{x \rightarrow 0} x$, to exist and be nonzero. This is not the case, so we cannot apply directly. \square

4.4. **Exercise 2.3.30.** Show that the Product Law cannot be used to evaluate $\lim_{x \rightarrow \pi/2} (x - \pi/2) \tan x$.

Solution. The Product Law requires the limit of each factor to exist. However, $\lim_{x \rightarrow \pi/2} \tan x$ does not exist. \square

4.5. **Exercise 2.3.31.** Give an example where $\lim_{x \rightarrow 0} (f(x) + g(x))$ exists but neither $\lim_{x \rightarrow 0} f(x)$ nor $\lim_{x \rightarrow 0} g(x)$ exists.

Solution. Let $f(x)$ be any function defined on a neighborhood of 0 (but not necessarily at 0) such that $\lim_{x \rightarrow 0} f(x)$ does not exist (e.g., $f(x) = 1/x$). Let $g(x) = -f(x)$. Then of course $\lim_{x \rightarrow 0} g(x)$ also does not exist (otherwise by the Constant Multiple Law, $\lim_{x \rightarrow 0} f(x)$ also exists). But notice $f(x) + g(x)$ is identically zero in a neighborhood of 0 (but not necessarily at 0). So $\lim_{x \rightarrow 0} (f(x) + g(x)) = 0$ exists. \square

4.6. **Exercise 2.3.32.** Assume that the limit $L_a = \lim_{x \rightarrow 0} \frac{a^x - 1}{x}$ exists and that $\lim_{x \rightarrow 0} a^x = 1$ for all $a > 0$. Prove that $L_{ab} = L_a + L_b$ for $a, b > 0$. [Hint: $(ab)^x - 1 = a^x(b^x - 1) + (a^x - 1)$.]

Solution. By definition, $L_{ab} = \lim_{x \rightarrow 0} \frac{(ab)^x - 1}{x} = \lim_{x \rightarrow 0} a^x \frac{b^x - 1}{x} + \frac{a^x - 1}{x}$. Since $\lim_{x \rightarrow 0} a^x = 1$ by assumption and $\lim_{x \rightarrow 0} \frac{b^x - 1}{x} = L_b$ exists by assumption, the Product Law states $\lim_{x \rightarrow 0} a^x \frac{b^x - 1}{x} = 1 \cdot L_b$. Now $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = L_a$ by assumption, so the Sum Law yields $\lim_{x \rightarrow 0} a^x \frac{b^x - 1}{x} + \frac{a^x - 1}{x} = L_b + L_a$. \square

4.7. **Exercise 2.3.38.** Assuming that $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$, which of the following statements is necessarily true?

- (a) $f(0) = 0$.
- (b) $\lim_{x \rightarrow 0} f(x) = 0$.

Solution. Remember that the value of $f(x)$ at $x = 0$ never matters when we evaluate the limit $\lim_{x \rightarrow 0} f(x)$. So (a) is not (necessarily) true.

Recall that $\lim_{x \rightarrow 0} x = 0$, so by the Product Law, $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x \cdot \frac{f(x)}{x} = \lim_{x \rightarrow 0} x \cdot \lim_{x \rightarrow 0} \frac{f(x)}{x} = 0 \cdot 1 = 0$. Since $\lim_{x \rightarrow 0} \frac{f(x)}{x} = 1$, and $\lim_{x \rightarrow 0} x = 0$, we get $\lim_{x \rightarrow 0} f(x) = 0$. \square