MATH 31A DISCUSSION

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1. Basic Limits

1.1. **Basic Limit Laws.** Assume that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then:

(a) Sum Law:
$$\lim_{x\to c} (f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

(b) Constant Multiple Law: For any number $k \in \mathbb{R}$,
$$\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x).$$

(c) Product Law:
$$\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right).$$

(d) Quotient Law: If $\lim_{x\to c} g(x) \neq 0$, then
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

1.2. **Exercise 2.3.22.** Evaluate the limit $\lim_{z\to 1} \frac{z^{-1}+z}{z+1}$.

Solution. Recall that $\lim_{z\to 1} z = 1$ and $\lim_{z\to 1} 1 = 1$. By the Quotient Law, $\lim_{z\to 1} z^{-1} = \frac{\lim_{z\to 1} 1}{\lim_{z\to 1} z} = \frac{1}{1} = 1$. By the Sum Law, $\lim_{z\to 1} z^{-1} + z = \lim_{z\to 1} z^{-1} + \lim_{z\to 1} z = 1 + 1 = 2$. By the Sum Law, $\lim_{z\to 1} z + 1 = 2$. So by the Quotient Law, $\lim_{z\to 1} \frac{z^{-1}+z}{z+1} = \frac{\lim_{z\to 1} z^{-1}+z}{\lim_{z\to 1} z+1} = \frac{2}{2} = 1$. □

1.3. **Exercise 2.3.29.** Can the Quotient Law be applied to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$?

Solution. The Quotient Law requires the limit of the denominator, namely, $\lim_{x\to 0} x$, to exist and be nonzero. This is not the case, so we cannot apply directly. □

1.4. **Exercise 2.3.30.** Show that the Product Law cannot be used to evaluate $\lim_{x\to \pi/2}(x - \pi/2) \tan x$.

Solution. The Product Law requires the limit of each factor to exist. However, $\lim_{x\to\pi/2} \tan x$ does not exist. □

1.5. **Exercise 2.3.31.** Give an example where $\lim_{x\to 0} (f(x) + g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

Solution. Let $f(x)$ be any function defined on a neighborhood of 0 (but not necessarily at 0) such that $\lim_{x\to 0} f(x)$ does not exist (e.g., $f(x) = 1/x$). Let $g(x) = -f(x)$. Then of course $\lim_{x\to 0} g(x)$ also does not exist (otherwise by the Constant Multiple Law, $\lim_{x\to 0} f(x)$ also exists). But notice $f(x) + g(x)$ is identicaly zero in a neighborhood of 0 (but not necessarily at 0). So $\lim_{x\to 0} (f(x) + g(x)) = 0$ exists. □

1.6. **Exercise 2.3.32.** Assume that the limit $L_a = \lim_{x\to 0} \frac{a^x-1}{x}$ exists and that $\lim_{x\to 0} a^x = 1$ for all $a > 0$. Prove that $L_{ab} = L_a + L_b$ for $a, b > 0$. [Hint: $(ab)^x - 1 = a^x(b^x - 1) + (a^x - 1)$.]

Solution. By definition, $L_{ab} = \lim_{x\to 0} \frac{(ab)^x-1}{x} = \lim_{x\to 0} a^x \frac{b^x-1}{x} + \frac{a^x-1}{x}$. Since $\lim_{x\to 0} a^x = 1$ by assumption and $\lim_{x\to 0} \frac{b^x-1}{x} = L_b$ exists by assumption, the Product Law states $\lim_{x\to 0} a^x \frac{b^x-1}{x} = 1 \cdot L_b$. Now $\lim_{x\to 0} \frac{a^x-1}{x} = L_a$ by assumption, so the Sum Law yields $\lim_{x\to 0} a^x \frac{b^x-1}{x} + \frac{a^x-1}{x} = L_b + L_a$. □

1.7. **Exercise 2.3.38.** Assuming that $\lim_{x\to 0} \frac{f(x)}{x} = 1$, which of the following statements is necessarily true?

(a) $f(0) = 0$.
(b) $\lim_{x\to 0} f(x) = 0$.

Solution. Remember that the value of $f(x)$ at $x = 0$ never matters when we evaluate the limit $\lim_{x\to 0} f(x)$. So (a) is not (necessarily) true.

Recall that $\lim_{x\to 0} x = 0$, so by the Product Law, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x \cdot \frac{f(x)}{x} = \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{f(x)}{x} = 0 \cdot 1 = 0$. Since $\lim_{x\to 0} \frac{f(x)}{x} = 1$, and $\lim_{x\to 0} x = 0$, we get $\lim_{x\to 0} f(x) = 0$. □