MATH 31A DISCUSSION

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1. Applications of the Derivative

1.1. **Basics.**

1.1.1. *Critical Points.* A number c in the domain of f is called a *critical point* if either $f'(c) = 0$ or $f'(c)$ does not exist.

1.1.2. *Local Extrema.* If $f(c)$ is a local extremum, then c is a critical point of f.

1.1.3. *Extrema on Closed Interval.* If $f(x)$ is continuous on $[a, b]$, and $f(c)$ be an extremum on $[a, b]$. Then c is either a critical point or one of the endpoints a or b.

1.1.4. *Rolle's Theorem.* Assume that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a number c between a and b such that $f'(c) = 0$.

1.1.5. *Mean Value Theorem.* Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In particular, if $f(a) = f(b)$, we get Rolle's Theorem.

1.2. **Exercise 4.2.39.** Find the maximum and minimum values of $y = \sin x \cos x$ on $[0, \frac{\pi}{2}]$.

Solution. Notice $y' = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$. If $y' = 0$, then $\cos^2 x = \frac{1}{2}$, so $\cos x = \pm\frac{\sqrt{2}}{2}$. In $[0, \frac{\pi}{2}]$, this occur at $x = \frac{\pi}{4}$. Now $f(0) = f(\frac{\pi}{2}) = 0$, and $f(\frac{\pi}{4}) = \frac{1}{2}$. So min is 0 and max is $\frac{1}{2}$. □

1.3. **Exercise 4.2.73–74.** Show that $f(x) = x^2 - 2x + 3$ takes on only positive values. Find conditions on r and s under which the quadratic function $f(x) = x^2 + rx + s$ takes on only positive values. Show that if f takes on both positive and negative values, then its minimum value occurs at the midpoint between the two roots.

Solution. For $f(x) = x^2 - 2x + 3$, we have $f'(x) = 2x - 2$, so $x = 1$ is critical point, and $f(1) = 2 > 0$. More generally, for $f(x) = x^2 + rx + s$, we have $f'(x) = 2x + r$, so $x = -\frac{r}{2}$ is critical point. Now $f(-\frac{r}{2}) = s - \frac{r^2}{4}$. So if we want this to be positive, we must have $s > \frac{r^2}{4}$. If f takes on both positive and negative values, then the roots are $x = \frac{-r \pm \sqrt{r^2 - 4s}}{2}$, whose midpoint is $x = -\frac{r}{2}$, as desired. □

1.4. **Exercise 4.3.42.** Show that $f(x) = x^3 - 2x^2 + 2x$ is an increasing function.

Solution. Notice $f'(x) = 3x^2 - 4x + 2$. What is its minimum? Find its critical points: $f''(x) = 6x - 4$, so $x = \frac{2}{3}$ is the critical point. So $f'(x)$ has its minimum at $x = \frac{2}{3}$, which is $f'(\frac{2}{3}) = \frac{2}{3}$. So $f'(x) > 0$, thus $f(x)$ is increasing. □

1.5. **Exercise 4.3.53–55.** Prove that if $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for $x \geq 0$, then $f(x) \leq g(x)$ for all $x \geq 0$. Prove the following:

(a) $\sin x \leq x$ for $x \geq 0$.
(b) $\cos x \geq 1 - \frac{1}{2}x^2$,
(c) $\sin x \geq x - \frac{1}{6}x^3$,
(d) $\cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Solution. Let $h(x) = f(x) - g(x)$. Notice $h'(x) = f'(x) - g'(x) \leq 0$. So $h(x)$ is non-increasing. Since $h(0) = 0$, we have that for $x \geq 0$, $h(0) \leq 0$. So $f(x) - g(x) \leq 0$, thus $f(x) \leq g(x)$, as desired.

Since $\sin x$ and x agree at $x = 0$, and the derivatives $\cos x \leq 1$ as required, we apply what we got above to conclude the desired result. The rest follows similarly. □