MATH 31A DISCUSSION

JED YANG

1. Theorems

1.1. **Area Between Graphs.** If $f(x) \geq g(x)$ on $[a, b]$, then the area between the graphs is $\int_a^b f(x) - g(x)\, dx$.

1.2. **Volume of a Solid.** If the cross section of a solid between $[a, b]$ has area $A(x)$, then the volume is $\int_a^b A(x)\, dx$.

1.3. **Solid of Revolution.** If a solid of revolution is formed with radii $r < R$ on $[a, b]$, then the volume is $\pi \int_a^b (R^2 - r^2)\, dx$.

2. Applications of the Integral

2.1. **Exercise 6.1.47.** Set up (but do not evaluate) an integral that expresses the area between the circles $x^2 + y^2 = 2$ and $x^2 + (y - 1)^2 = 1$.

Solution. Solving $x^2 + y^2 = 2$ and $x^2 + (y-1)^2 = 1$, we get $2 - y^2 + y^2 - 2y + 1 = 1$, that is $y = 1$ and $x = \pm 1$. So the region is between $x = -1$ and $x = 1$, above $y = 1 - \sqrt{1 - x^2}$ and below $y = \sqrt{2 - x^2}$. Thus an integral representing the region is $\int_{-1}^1 \sqrt{2 - x^2} - (1 - \sqrt{1 - x^2})\, dx$. (The area is $\pi - 1$.) □

2.2. **Exercise 6.2.21.** Let S be the solid obtained by intersecting two cylinders of radius r whose axes are perependicular. Find the volume of S as a function of r.

Solution. The horizontal cross section of each cylinder at distance y from the central axis is a rectangular strip. The strip's width w satisfies the Pythagorean relationship $(w/2)^2 + y^2 = r^2$, hence $w = 2\sqrt{r^2 - y^2}$.

The area of the horizontal cross section of S at distance y is thus $A = w^2 = 4(r^2 - y^2)$.

Finally, the volume of S is thus $V = \int_{-r}^r 4(r^2 - y^2)\, dy = 4\left[r^2 y - y^3/3\right]_{-r}^r = 8(r^3 - r^3/3) = \frac{16}{3} r^3$. □

2.3. **Exercise 6.3.36.** Find volume of the solid obtained by rotating the region enclosed by the graphs $y = x^2$, $y = 12 - x$, and $x = 0$ about the axis $y = 15$.

Solution. Solving $x^2 = 12 - x$ gives $x = 3$ or -4. We'll take the $x \geq 0$ portion, since the $x \leq 0$ portion intersects the axis and is weird. (N.B., this problem is not very clearly written.) The distances are $15 - x^2$ and $15 - 12 + x = 3 + x$, respectively. Thus the volume is given by $V = \pi \int_0^3 (15 - x^2)^2 - (3 + x)^2\, dx = \pi \left[x^5/5 - 31x^3/3 - 3x^2 + 216x\right]_0^3 = 1953\pi/5$. □

2.4. **Exercise 6.3.54.** Verify the formula

$$\int_{x_1}^{x_2} (x - x_1)(x - x_2)\,dx = \frac{1}{6}(x_1 - x_2)^3.$$

Then prove that the solid obtained by rotating the shaded region above $y = mx + c$ and below $y^2 = ax + b$ about the x-axis has volume $V = \frac{\pi}{6}BH^2$, with B the base of the region and H the height.

Solution. The verification is a trivial process, and is left as an exercise to the student. Let x_1 and x_2 be the roots of $f(x) = ax + b - (mx + c)^2$, where $x_1 < x_2$. Notice that $V = \pi \int_{x_1}^{x_2} f(x)\,dx$. Since x_1 and x_2 are roots, and the coefficient of x^2 is $-m^2$, we get that $f(x) = -m^2(x - x_1)(x - x_2)$. Thus the integral becomes $V = \pi \int_{x_1}^{x_2} (-m^2)(x - x_1)(x - x_2)\,dx$. By the formula given, we get $V = -m^2\pi \cdot \frac{1}{6}(x_1 - x_2)^3$. Notice that $x_2 - x_1 = B$ and $m = \frac{H}{B}$, we finally get $V = \frac{\pi}{6}BH^2$, as desired. □