MATH 31A DISCUSSION

JED YANG

1. Introduction

Lecture 1

- Instructor: Steve Butler.
- Location: HAINES 39.

Sections 1A and 1B

- Email: `mailto:jedyang@ucla.edu`.
- Office: MS 6617A.
- Office hours: R 12:30–13:30.
- Discussion Location: MS 5117 (T) and 5138 (R).
- Website: `http://www.math.ucla.edu/~jedyang/31a.1.10w/`.
- SMC: Jan. 6–Mar. 11, M–R 09:00–15:00, MS 3974, T 12:00–13:00.

2. Administration

- HW due Fridays in lecture, can turn in early to me, and I will hand back in section.
- Textbook: Rogawski, Single Variable Calculus, 2008.
- Confirm office hour.

3. Precalculus Review

3.1. **Exercise 1.2.21.** Find the equation of the perpendicular bisector of the segment joining $(1, 2)$ and $(5, 4)$.

Solution. Slope of segment is $m_1 = \frac{4-2}{5-1} = \frac{1}{2}$. Slope of perpendicular bisector is $m_2 = -1/m_1 = -2$. Mid point is $(\frac{1+5}{2}, \frac{2+4}{2})$. So the equation can be written as $y - 3 = -2(x - 3)$. □

3.2. **Exercise 1.2.23.** Find the equation of the line with x-intercept $x = 4$ and y-intercept $y = 3$.

Solution. Equation of the line is $y = mx+b$, where b is the y-intercept, hence $b = 3$. The x-intercept $x = 4$ will yield $y = 0$ (by definition), so substituting, we may solve for m. We get $0 = 4m + 3$, hence $m = -\frac{3}{4}$. So the equation can be written as $y = -\frac{3}{4}x + 3$. □

3.3. **Exercise 1.2.24.** A line of slope $m = 2$ passes through $(1, 4)$. Find y such that $(3, y)$ lies on the line.

Solution. One way is to write down an equation of the line in point-slope form: $y = 2(x-1)+4$. Then we see clearly that if $x = 3$, then $y = 8$. Alternatively, the slope m is the change of y over the change of x. Symbolically, $m = \frac{\Delta y}{\Delta x}$, or $\Delta y = m\Delta x$. This concept will be useful later when we deal with differentials $dy = m\,dx$. Since the change in x is $\Delta x = 3-1 = 2$, we get that the change in y is $\Delta y = y-4 = 2\cdot 2 = 4$, hence $y = 8$. This method seems longer, but conceptually it is easier to do in one's head, and will lead to intution for calculus later. □

3.4. **Exercise 1.4.55.** Use the addition formulae for sine and cosine to prove

$$\tan(a+b) = \frac{\tan a + \tan b}{1-\tan a \tan b} \quad (1)$$

$$\cot(a-b) = \frac{\cot a \cot b + 1}{\cot b - \cot a}. \quad (2)$$

Proof. Recall that

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \quad (3)$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b. \quad (4)$$

Now

$$\tan(a+b) = \frac{\sin(a+b)}{\cos(a+b)} \quad (5)$$

$$= \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b} \quad (6)$$

$$= \frac{\frac{\sin a}{\cos a} + \frac{\sin b}{\cos b}}{1 - \frac{\sin a}{\cos a}\frac{\sin b}{\cos b}} \quad (7)$$

$$= \frac{\tan a + \tan b}{1-\tan a \tan b}, \quad (8)$$

where we get from (6) to (7) by dividing top and bottom by $\cos a \cos b$.

The case for cotangent is completely analogous. Remember $\cot x = \frac{\cos x}{\sin x}$ and that $\sin(-b) = -\sin(b)$ and $\cos(-b) = \cos(b)$. Work out the details and convince yourself. □

3.5. **Exercise 1.4.56.** Let θ be the angle between the line $y = mx+b$ and the x-axis. Prove that $m = \tan\theta$.

Proof. This is trivial. □

3.6. **Exercise 1.4.57.** Let L_1 and L_2 be the lines of slope m_1 and m_2, respectively. Show that the angle θ between L_1 and L_2 satisfies $\cot\theta = \frac{m_2 m_1 + 1}{m_2 - m_1}$.

Proof. This is immediate by using Exercises 55 and 56. □

3.7. **Exercise 1.4.58. Perpendicular Lines.** Use Exercise 57 to prove that two lines with nonzero slopes m_1 and m_2 are perpendicular if and only if $m_2 = -1/m_1$.

Proof. What is $\cot(\pi/2)$? □

3.8. **Exercise 1.4.59.** Apply the double-angle formula to prove:

(a) $\cos\frac{\pi}{8} = \frac{1}{2}\sqrt{2+\sqrt{2}}$.

(b) $\cos\frac{\pi}{16} = \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$.

Guess the values of $\cos\frac{\pi}{32}$ and of $\cos\frac{\pi}{2^n}$ for all n.

Proof. Recall $\cos^2 t = \frac{1+\cos(2t)}{2}$. For the general case, let $a_0 = 0$ and define inductively $a_n = \sqrt{2+a_{n-1}}$. We claim that for $n \geq 1$, we have $\cos\frac{\pi}{2^n} = \frac{1}{2}a_{n-1}$. The base case is trivial. By induction, assume $\cos\frac{\pi}{2^n} = \frac{1}{2}a_{n-1}$. By the half-angle formula, we get $\cos\frac{\pi}{2^{n+1}} = \sqrt{\frac{1}{2}(1+\frac{1}{2}a_{n-1})} = \sqrt{\frac{1}{4}(2+a_{n-1})} = \frac{1}{2}a_n$. □

4. Basic Limits

4.1. **Basic Limit Laws.** Assume that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then:

(a) Sum Law:

$$\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

(b) Constant Multiple Law: For any number $k \in \mathbb{R}$,

$$\lim_{x\to c} kf(x) = k\lim_{x\to c} f(x).$$

(c) Product Law:

$$\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right).$$

(d) Quotient Law: If $\lim_{x\to c} g(x) \neq 0$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

4.2. **Exercise 2.3.22.** Evaluate the limit $\lim_{z\to 1}\frac{z^{-1}+z}{z+1}$.

Solution. Recall that $\lim_{z\to 1} z = 1$ and $\lim_{z\to 1} 1 = 1$. By the Quotient Law, $\lim_{z\to 1} z^{-1} = \frac{\lim_{z\to 1} 1}{\lim_{z\to 1} z} = \frac{1}{1} = 1$. By the Sum Law, $\lim_{z\to 1} z^{-1} + z = \lim_{z\to 1} z^{-1} + \lim_{z\to 1} z = 1+1 = 2$. By the Sum Law, $\lim_{z\to 1} z+1 = 2$. So by the Quotient Law, $\lim_{z\to 1}\frac{z^{-1}+z}{z+1} = \frac{\lim_{z\to 1} z^{-1}+z}{\lim_{z\to 1} z+1} = \frac{2}{2} = 1$. □

4.3. **Exercise 2.3.29.** Can the Quotient Law be applied to evaluate $\lim_{x\to 0}\frac{\sin x}{x}$?

Solution. The Quotient Law requires the limit of the denominator, namely, $\lim_{x\to 0} x$, to exist and be nonzero. This is not the case, so we cannot apply directly. □

4.4. **Exercise 2.3.30.** Show that the Product Law cannot be used to evaluate $\lim_{x\to\pi/2}(x-\pi/2)\tan x$.

Solution. The Product Law requires the limit of each factor to exist. However, $\lim_{x\to\pi/2}\tan x$ does not exist. □

4.5. **Exercise 2.3.31.** Give an example where $\lim_{x\to 0}(f(x)+g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

Solution. Let $f(x)$ be any function defined on a neighborhood of 0 (but not necessarily at 0) such that $\lim_{x\to 0} f(x)$ does not exist (e.g., $f(x) = 1/x$). Let $g(x) = -f(x)$. Then of course $\lim_{x\to 0} g(x)$ also does not exist (otherwise by the Constant Multiple Law, $\lim_{x\to 0} f(x)$ also exists). But notice $f(x) + g(x)$ is identicaly zero in a neighborhood of 0 (but not necessarily at 0). So $\lim_{x\to 0} (f(x) + g(x)) = 0$ exists. □

4.6. **Exercise 2.3.32.** Assume that the limit $L_a = \lim_{x\to 0} \frac{a^x-1}{x}$ exists and that $\lim_{x\to 0} a^x = 1$ for all $a > 0$. Prove that $L_{ab} = L_a + L_b$ for $a, b > 0$. [Hint: $(ab)^x - 1 = a^x(b^x - 1) + (a^x - 1)$.]

Solution. By definition, $L_{ab} = \lim_{x\to 0} \frac{(ab)^x-1}{x} = \lim_{x\to 0} a^x \frac{b^x-1}{x} + \frac{a^x-1}{x}$. Since $\lim_{x\to 0} a^x = 1$ by assumption and $\lim_{x\to 0} \frac{b^x-1}{x} = L_b$ exists by assumption, the Product Law states $\lim_{x\to 0} a^x \frac{b^x-1}{x} = 1 \cdot L_b$. Now $\lim_{x\to 0} \frac{a^x-1}{x} = L_a$ by assumption, so the Sum Law yields $\lim_{x\to 0} a^x \frac{b^x-1}{x} + \frac{a^x-1}{x} = L_b + L_a$. □

4.7. **Exercise 2.3.38.** Assuming that $\lim_{x\to 0} \frac{f(x)}{x} = 1$, which of the following statements is necessarily true?

(a) $f(0) = 0$.
(b) $\lim_{x\to 0} f(x) = 0$.

Solution. Remember that the value of $f(x)$ at $x = 0$ never matters when we evaluate the limit $\lim_{x\to 0} f(x)$. So (a) is not (necessarily) true.

Recall that $\lim_{x\to 0} x = 0$, so by the Product Law, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x \cdot \frac{f(x)}{x} = \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{f(x)}{x} = 0 \cdot 1 = 0$. Since $\lim_{x\to 0} \frac{f(x)}{x} = 1$, and $\lim_{x\to 0} x = 0$, we get $\lim_{x\to 0} f(x) = 0$. □