MATH 31A DISCUSSION

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1. Limits

1.1. **Basic Limit Laws.** Assume that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then:

(a) Sum Law:
$$\lim_{x\to c}\left(f(x)+g(x)\right) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x).$$

(b) Constant Multiple Law: For any number $k \in \mathbb{R}$,
$$\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x).$$

(c) Product Law:
$$\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right).$$

(d) Quotient Law: If $\lim_{x\to c} g(x) \neq 0$, then
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}.$$

1.2. **Exercise 2.3.22.** Evaluate the limit $\lim_{z\to 1} \frac{z^{-1}+z}{z+1}$.

Solution. Recall that $\lim_{z\to 1} z = 1$ and $\lim_{z\to 1} 1 = 1$. By the Quotient Law, $\lim_{z\to 1} z^{-1} = \frac{\lim_{z\to 1} 1}{\lim_{z\to 1} z} = \frac{1}{1} = 1$. By the Sum Law, $\lim_{z\to 1} z^{-1} + z = \lim_{z\to 1} z^{-1} + \lim_{z\to 1} z = 1+1 = 2$. By the Sum Law, $\lim_{z\to 1} z+1 = 2$. So by the Quotient Law, $\lim_{z\to 1} \frac{z^{-1}+z}{z+1} = \frac{\lim_{z\to 1} z^{-1}+z}{\lim_{z\to 1} z+1} = \frac{2}{2} = 1$. □

1.3. **Exercise 2.3.29.** Can the Quotient Law be applied to evaluate $\lim_{x\to 0} \frac{\sin x}{x}$?

Solution. The Quotient Law requires the limit of the denominator, namely, $\lim_{x\to 0} x$, to exist and be nonzero. This is not the case, so we cannot apply directly. □

1.4. **Exercise 2.3.30.** Show that the Product Law cannot be used to evaluate $\lim_{x\to \pi/2}(x-\pi/2)\tan x$.

Solution. The Product Law requires the limit of each factor to exist. However, $\lim_{x\to \pi/2} \tan x$ does not exist. □

1.5. **Exercise 2.3.31.** Give an example where $\lim_{x\to 0}(f(x)+g(x))$ exists but neither $\lim_{x\to 0} f(x)$ nor $\lim_{x\to 0} g(x)$ exists.

Solution. Let $f(x)$ be any function defined on a neighborhood of 0 (but not necessarily at 0) such that $\lim_{x\to 0} f(x)$ does not exist (e.g., $f(x) = 1/x$). Let $g(x) = -f(x)$. Then of course $\lim_{x\to 0} g(x)$ also does not exist (otherwise by the Constant Multiple Law, $\lim_{x\to 0} f(x)$ also exists). But notice $f(x)+g(x)$ is identicaly zero in a neighborhood of 0 (but not necessarily at 0). So $\lim_{x\to 0}(f(x)+g(x)) = 0$ exists. □

1.6. **Exercise 2.3.32.** Assume that the limit $L_a = \lim_{x\to 0} \frac{a^x - 1}{x}$ exists and that $\lim_{x\to 0} a^x = 1$ for all $a > 0$. Prove that $L_{ab} = L_a + L_b$ for $a, b > 0$. [Hint: $(ab)^x - 1 = a^x(b^x - 1) + (a^x - 1)$.]

Solution. By definition, $L_{ab} = \lim_{x\to 0} \frac{(ab)^x - 1}{x} = \lim_{x\to 0} a^x \frac{b^x-1}{x} + \frac{a^x-1}{x}$. Since $\lim_{x\to 0} a^x = 1$ by assumption and $\lim_{x\to 0} \frac{b^x-1}{x} = L_b$ exists by assumption, the Product Law states $\lim_{x\to 0} a^x \frac{b^x-1}{x} = 1 \cdot L_b$. Now $\lim_{x\to 0} \frac{a^x-1}{x} = L_a$ by assumption, so the Sum Law yields $\lim_{x\to 0} a^x \frac{b^x-1}{x} + \frac{a^x-1}{x} = L_b + L_a$. □

1.7. **Exercise 2.3.38.** Assuming that $\lim_{x\to 0} \frac{f(x)}{x} = 1$, which of the following statements is necessarily true?

(a) $f(0) = 0$.
(b) $\lim_{x\to 0} f(x) = 0$.

Solution. Remember that the value of $f(x)$ at $x = 0$ never matters when we evaluate the limit $\lim_{x\to 0} f(x)$. So (a) is not (necessarily) true.

Recall that $\lim_{x\to 0} x = 0$, so by the Product Law, $\lim_{x\to 0} f(x) = \lim_{x\to 0} x \cdot \frac{f(x)}{x} = \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{f(x)}{x} = 0 \cdot 1 = 0$. Since $\lim_{x\to 0} \frac{f(x)}{x} = 1$, and $\lim_{x\to 0} x = 0$, we get $\lim_{x\to 0} f(x) = 0$. □

1.8. **Exercise 2.4.31.** Determine the points at which the function

$$f(x) = \tan(\sin x)$$

is discontinuous and state the type of discontinuity: removable, jump, infinite, or none of these.

Solution. Recall that $\tan x$ is discontinuous at $x = k\pi/2$ for odd k. However, we have $-1 \le \sin x \le 1$, and $\tan x$ is continuous on $[-1, 1]$, so $\tan(\sin x)$ is continuous everywhere. □

1.9. **Exercise 2.4.48. Sawtooth Function.** Draw the graph of $f(x) = x - [x]$. At which points is f discontinuous? Is it left- or right-continuous at those points?

Solution. Recall that $[x]$ is the floor function, defined as the greatest integer smaller than or equal to x. The function f is discontinuous at the integers $\mathbb{Z}$, but is right-continuous everywhere. In particular, at the discontinuities, f is right-continuous but not left-continuous. □

1.10. **Exercise 2.5.21.** Evaluate the limit

$$\lim_{x\to 2} \frac{x-2}{\sqrt{x} - \sqrt{4-x}}.$$

Solution. Multiply by the conjugate $\sqrt{x} + \sqrt{4-x}$ for both the numerator and denominator, we get

$$\begin{aligned}
\frac{x-2}{\sqrt{x}-\sqrt{4-x}} &= \frac{(x-2)(\sqrt{x}+\sqrt{4-x})}{x-(4-x)} && (1)\\
&= \frac{(x-2)(\sqrt{x}+\sqrt{4-x})}{2(x-2)} && (2)\\
&= \frac{\sqrt{x}+\sqrt{4-x}}{2} && (3)
\end{aligned}$$

for $x \neq 2$. Since the limit as x approaches 2 does not depend on the value at 2, we get

$$\lim_{x \to 2} \frac{x-2}{\sqrt{x} - \sqrt{4-x}} = \lim_{x \to 2} \frac{\sqrt{x} + \sqrt{4-x}}{2} = \frac{\sqrt{2} + \sqrt{4-2}}{2} = \sqrt{2},$$

where we can substitute 2 for x since the transformed function is continuous at $x = 2$. $\square$

1.11. **Exercise 2.5.37.** Evaluate the limit

$$\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^3 - 1}.$$

Solution. First factor $x^2 - 3x + 2 = (x-1)(x-2)$ and $x^3 - 1 = (x-1)(x^2 + x + 1)$. Thus

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 - 3x + 2}{x^3 - 1} &= \lim_{x \to 1} \frac{(x-1)(x-2)}{(x-1)(x^2 + x + 1)} && (4) \\ &= \lim_{x \to 1} \frac{x-2}{x^2 + x + 1} = -\frac{1}{3}. && (5) \end{aligned}$$

$\square$

1.12. **Squeeze Theorem.** Assume that for $x \neq c$ (in some open interval containing c), we have

$$\ell(x) \leq f(x) \leq u(x) \quad \text{and} \quad \lim_{x \to c} \ell(x) = \lim_{x \to c} u(x) = L.$$

Then $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = L$.

1.13. **Exercise 2.6.46.** Use the Squeeze Theorem to prove that if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$.

Proof. Notice $-|f(x)| \leq f(x) \leq |f(x)|$. Furthermore,

$$\lim_{x \to c} -|f(x)| = -\lim_{x \to c} |f(x)| = 0 = \lim_{x \to c} |f(x)|.$$

Thus by the Squeeze Theorem, $\lim_{x \to c} f(x) = 0$. $\square$

1.14. **Exercise 2.6.51.** Prove

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0.$$

[Hint: Using a diagram of the unit circle and the Pythagorean Theorem, show that

$$\sin^2\theta \leq (1 - \cos\theta)^2 + \sin^2\theta \leq \theta^2.$$

Conclude that $\sin^2\theta \leq 2(1 - \cos\theta) \leq \theta^2$.]

Proof. The first inequality is obvious since a square is non-negative. The middle and the right hand side are precisely the squares of the lengths of the secant line and the arc that subtends the given angle, respectively. Expanding and recalling the trignometric identity $\sin^2\theta + \cos^2\theta = 1$, we get

$$(1-\cos\theta)^2 + \sin^2\theta = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta = 2 - 2\cos\theta.$$

Therefore we get

$$\sin^2\theta \le 2(1-\cos\theta) \le \theta^2.$$

Dividing each side by 2θ, we get

$$\frac{\sin^2\theta}{2\theta} \le \frac{1-\cos\theta}{\theta} \le \frac{\theta}{2},$$

if $\theta > 0$. Notice the limit of the left and the right hand sides are both zero as $\theta \to 0$ Thus by the Squeeze Theorem, we get

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0,$$

as desired. If $\theta < 0$, the inequalities will be switched, but the result (and analysis) still holds.

Alternatively, we can consider

$$\frac{1-\cos\theta}{\theta} = \frac{1-\cos^2\theta}{\theta} \cdot \frac{1}{1+\cos\theta}.$$

The first factor is $\frac{\sin^2\theta}{\theta}$ which approaches 0 as $\theta \to 0$. The second factor approaches $\frac{1}{2}$ as $\theta \to 0$. So by the Product Law, the limit is 0 as $\theta \to 0$. $\square$