# MATH 31A DISCUSSION

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## More Applications of the Derivative

### 1. MVT and Monotonicity

1.1. **Mean Value Theorem.** Assume that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Then there exists a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In particular, if $f(a) = f(b)$, we get Rolle's Theorem.

1.2. **Exercise 4.3.42.** Show that $f(x) = x^3 - 2x^2 + 2x$ is an increasing function.

*Solution.* Notice $f'(x) = 3x^2 - 4x + 2$. What is its minimum? Find its critical points: $f''(x) = 6x - 4$, so $x = \frac{2}{3}$ is the critical point. So $f'(x)$ has its minimum at $x = \frac{2}{3}$, which is $f'(\frac{2}{3}) = \frac{2}{3}$. So $f'(x) > 0$, thus $f(x)$ is increasing. □

1.3. **Exercise 4.3.53–55.** Prove that if $f(0) = g(0)$ and $f'(x) \leq g'(x)$ for $x \geq 0$, then $f(x) \leq g(x)$ for all $x \geq 0$. Prove the following:
(a) $\sin x \leq x$ for $x \geq 0$.
(b) $\cos x \geq 1 - \frac{1}{2}x^2$,
(c) $\sin x \geq x - \frac{1}{6}x^3$,
(d) $\cos x \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

*Solution.* Let $h(x) = f(x) - g(x)$. Notice $h'(x) = f'(x) - g'(x) \leq 0$. So $h(x)$ is non-increasing. Since $h(0) = 0$, we have that for $x \geq 0$, $h(0) \leq 0$. So $f(x) - g(x) \leq 0$, thus $f(x) \leq g(x)$, as desired.

Since $\sin x$ and $x$ agree at $x = 0$, and the derivatives $\cos x \leq 1$ as required, we apply what we got above to get the desired result. The rest follows similarly. □

### 2. Graphs

2.1. **Basics.**

2.1.1. *Concavity.* If $f'(x)$ is increasing (or $f''(x) > 0$), then $f$ is *concave up* at $x$. If $f'(x)$ is decreasing (or $f''(x) < 0$), then $f$ is *concave down* at $x$.

2.1.2. *Inflection.* If $f''(c) = 0$ and $f''(x)$ changes sign at $x = c$, then $f(x)$ has a point of inflection at $x = c$.

2.1.3. *Second Derivative Test.* Let $f$ be differentiable and $c$ a critical point.
(a) If $f''(c) > 0$ then $f(c)$ is a local minimum.
(b) If $f''(c) < 0$ then $f(c)$ is a local maximum.
(c) If $f''(c) = 0$ then it is inconclusive, $f(c)$ may be a local min, max, or neither.

2.2. **Exercise 4.4.24.** Find the critical points of $f(x) = \sin^2 x + \cos x$, $x$ in $[0, \pi]$, and use the Second Derivative Test to determine whether each corresponds to a local minimum or maximum.

*Solution.* Notice $f'(x) = 2\sin x \cos x - \sin x$. Setting $f'(x) = 0$ and solving, we get $\sin x(2\cos x - 1) = 0$, so $x = 0, \pi/3, \pi$. Now $f''(x) = 2\cos^2 x - 2\sin^2 x - \cos x$. So $f''(0) = 1$, $f''(\frac{\pi}{3}) = -\frac{3}{2}$, $f''(\pi) = 3$, yielding local minima at $x = 0, \pi$ and maximum at $x = \pi/3$. □

2.3. **Exercise 4.4.53.** If $f'(c) = 0$ and $f(c)$ is neither a local min or max, must $x = c$ be a point of inflection? This is true of most "reasonable" examples, but it is not true in general. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

- Use the limit definition of the derivative to show that $f'(0)$ exists and $f'(0) = 0$.
- Show that $f(0)$ is neither a local min nor max.
- Show that $f'(x)$ changes sign infinitely often near $x = 0$ and conclude that $f(x)$ does not have a point of inflection at $x = 0$.

*Solution.* Recall Exercise 3.7.92 from 10/20.

Recall $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. So by definition, $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} h \sin \frac{1}{h}$. Using Squeeze Theorem and $-|h| \leq h \sin \frac{1}{h} \leq |h|$, we get that $f'(0) = 0$. Away from $x = 0$, we can use the formula and get $f'(x) = 2x \sin \frac{1}{x} + x^2 \cos \frac{1}{x} \cdot (-1) \frac{1}{x^2} = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$. Now $\lim_{x\to 0} f'(x)$ does not exist since $\lim_{x\to 0} 2x \sin \frac{1}{x} = 0$ by Squeeze Theorem but $\lim_{x\to 0} \cos \frac{1}{x}$ does not exist. □

2.4. **Bonus Question.** Assume $f''(x)$ exists and $f''(x) > 0$ for all $x$. Show that $f(x)$ cannot be always negative.

*Solution.* If $f'(x) \equiv 0$, then $f''(x) \equiv 0$, a contradiction. So there exists $b$ such that $f'(b) \neq 0$. Consider the tangent line at $x = b$ to $f(x)$. It is given by the equation $y = f'(b)(x - b) + f(b)$. Consider $g(x) = f(x) - f'(b)(x - b) - f(b)$. Notice that $g'(x) = f'(x) - f'(b)$ and $g''(x) = f''(x)$. So $g(b) = g'(b) = 0$, and $g''(x) > 0$ for all $x$. Hence $g'(x)$ is increasing. In particular, $g'(x) < 0$ for $x < b$ and $g'(x) > 0$ for $x > b$. If $x > b$, then by MVT, we get

$$\frac{g(x) - g(b)}{x - b} = g'(c)$$

for some $c$ in the interval $(b, x)$. In otherwords, since $c > b$, we have that $g'(c) > 0$ and $x - b > 0$, hence $g(x) - g(b) > 0$. Similarly, if $x < b$, we get $g'(c) < 0$, $x - b < 0$, so $g(x) - g(b) > 0$ as well. We thus conclude $g(x) \geq g(b)$ for all $x$. So $f(x) \geq f'(b)(x - b) + f(b)$ for all $x$. Since $f'(b) \neq 0$, there exists $x$ far enough from the origin such that $f'(b)(x - b) + f(b) > 0$, as desired. □