MATH 31A DISCUSSION

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1. Optimisation

1.1. **Exercise 4.6.15.** Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius r.

Solution. Let x and y be the side lengths. By Pythagorean theorem, they satisfy $x^2 + y^2 = 4r^2$. We want to maximize $A = xy$. Solve for $y = \sqrt{4r^2 - x^2}$. So $A = x\sqrt{4r^2 - x^2}$. We can maximize $A^2 = x^2(4r^2 - x^2) = 4r^2x^2 - x^4$. Differentiate to get $(A^2)' = 8r^2x - 4x^3$. Critical points are $8r^2x - 4x^3 = 0$ so $x = 0$ or $x^2 = 2r^2$, that is, $x = r\sqrt{2}$. End points $x = 0, 2r$. We get $A(0) = A(2r) = 0$ and $A(r\sqrt{2}) = 2r^2$. So area is maximized when $x = y = r\sqrt{2}$, that is, we have a square. □

1.2. **Catching a bus.** I am at one corner of a rectangular park with sides 60 and 300 metres. The bus stop is at the opposite corner. I can run on the grass at 5 m/s, and I can skateboard on the sidewalk (along the long side) at 13 m/s. I want to get to the bus stop by running across the park and then skateboarding on the sidewalk. Where should I run to in order to get there the fastest?

Solution. Let x be the distance that I skip on the long sidewalk. In otherwords, I run $\sqrt{60^2 + x^2}$ on grass, and then run on sidewalk for $300 - x$. Dividing by the rates, we get the total travel time is $y = \frac{300-x}{13} + \frac{\sqrt{60^2+x^2}}{5}$. Differentiating, we get $y' = -\frac{1}{13} + \frac{x}{5\sqrt{60^2+x^2}}$. Setting equal to 0, we solve and get $x = 25$.

Note that if there were a sidewalk on the short side too, then it would be faster to skate along the sidewalk for the entire time instead. Moral: Sometimes it is not good to cut corners. □

2. Basic Integration

2.1. **Exercise 4.8.58.** Let $f''(t) = t - \cos t$, $f'(0) = 2$, $f(0) = -2$. Find f' and f.

Solution. By integrating, we see $f'(t) = t^2/2 - \sin t + c$. Since $f'(0) = c = 2$, we get $f'(t) = t^2/2 - \sin t + 2$. Integrating again to get $f(t) = t^3/6 + \cos t + 2t + d$. Since $f(0) = 1 + d = -2$, we get $d = -3$, yielding $f(t) = t^3/y + \cos t + 2t - 3$. □