MATH 31A DISCUSSION

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1. Theorems

1.1. **Comparison Theorem.** If $g(x) \leq f(x)$ on an interval $[a, b]$, then

$$\int_a^b g(x)\,dx \leq \int_a^b f(x)\,dx.$$

1.2. **Fundamental Theorem of Calculus, I.** Assume that $f(x)$ is continuous on $[a, b]$ and let $F(x)$ be an antiderivative of $f(x)$ on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

1.3. **Fundamental Theorem of Calculus, II.** Assume that $f(x)$ is continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t)\,dt.$$

Then A is an antiderivative of f, that is, $A'(x) = f(x)$, or equivalently

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Furthermore, $A(x)$ satisfies the initial condition $A(a) = 0$.

2. Fundamental Theorem of Calculus

2.1. **Exercise 5.3.39.** Write the integral $\int_0^\pi |\cos x|\ dx$ as a sum of integrals without absolute values and evaluate.

Solution. Notice that $\cos x$ is nonnegative on $[0, \pi/2]$ and nonpositive on $[\pi/2, \pi]$. As such, the integral in question is $\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi/4} -\cos x\,dx = \sin x\ |_0^{\pi/2} - \sin x\ |_{\pi/2}^{\pi} = 1 - 0 - 0 + 1 = 2$. □

2.2. **Exercise 5.3.52.** Apply the Comparison Theorem to the inequality $\sin x \leq x$ (valid for $x \geq 0$) to prove $1 - \frac{x^2}{2} \leq \cos x \leq 1$. Apply it again to prove $x - \frac{x^3}{6} \leq \sin x \leq x$ (for $x \geq 0$).

Solution. On $[0, t]$ for some $t > 0$, we have $\sin x \leq x$. By the Comparison Theorem, we get $\int_0^t \sin x\,dx \leq \int_0^t x\,dx$. This gives $-\cos t + 1 \leq t^2/2$ hence $1 - \frac{t^2}{2} \leq \cos t$. Since cos is even, $\cos -t = \cos t$ satisfies the same inequality. Applying this again we get $\int_0^t 1 - \frac{x^2}{2}\,dx \leq \int_0^t \cos x\,dx$, yielding $t - \frac{t^3}{3} \leq \sin t$, as desired. □

2.3. **Exercise 5.4.40.** Find the smallest positive critical point of

$$F(x) = \int_0^x \cos(t^{3/2})\,dt$$

and determine whether it is a local min or max.

Solution. As usual, to find critical point, we take derivative and set to 0. By FTC2, we get $F'(x) = \cos(x^{3/2})$. The smallest positive zero is when $x^{3/2} = \frac{\pi}{2}$. So $x = (\pi/2)^{2/3}$ is the smallest positive critical point. Since F' goes from positive to negative, it is a local max. $\square$