MATH 31A DISCUSSION

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1. Theorems

1.1. The Substitution Method. If $F'(x) = f(x)$, then

$$\int f(u(x))u'(x)\,dx = F(u(x)) + C.$$

1.2. Area Between Graphs. If $f(x) \geq g(x)$ on $[a, b]$, then the area between the graphs is $\int_a^b f(x) - g(x)\,dx$.

1.3. Volume of a Solid. If the cross section of a solid between $[a, b]$ has area $A(x)$, then the volume is $\int_a^b A(x)\,dx$.

1.4. Solid of Revolution. If a solid of revolution is formed with radii $r < R$ on $[a, b]$, then the volume is $\pi \int_a^b (R^2 - r^2)\,dx$.

2. More Topics in Integration

2.1. Exercise 5.6.50. Evaluate the indefinite integral

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx.$$

Solution. Let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}\,dx$. So $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \int 2\cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$. $\square$

2.2. Exercise 5.6.54. Can They Both Be Right? Use $u = \tan x$ and $u = \sec x$ to evaluate

$$\int \tan x \sec^2 x\,dx.$$

Show that these yield different answers and explain the apparent contradiction.

Solution. If $u = \tan x$ then $du = \sec^2 x\,dx$, so the integral becomes $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\tan^2 x + C$. On the other hand, if $u = \sec x$ then $du = \tan x \sec x\,dx$, so the integral becomes $\int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sec^2 x + C$. Notice however that $\sec^2 x = \tan^2 x + 1$. $\square$

2.3. Exercise 5.6.72. Evaluate

$$\int_0^2 r\sqrt{5 - \sqrt{4 - r^2}}\,dr.$$

Solution. Let $u = 5 - \sqrt{4 - r^2}$, then $du = \frac{r}{\sqrt{4-r^2}}\,dr = \frac{r}{5-u}\,dr$. So the integral becomes $\int_3^5 (5 - u)\sqrt{u}\,du = \int_3^5 5u^{1/2} - u^{3/2}\,du = \frac{10}{3}u^{3/2} - \frac{2}{5}u^{5/2}\Big|_3^5 = \frac{20\sqrt{5}}{3} - \frac{32\sqrt{3}}{5}$. $\square$

2.4. **Exercise 6.1.47.** Set up (but do not evaluate) an integral that expresses the area between the circles $x^2 + y^2 = 2$ and $x^2 + (y-1)^2 = 1$.

Solution. Solving $x^2+y^2 = 2$ and $x^2+(y-1)^2 = 1$, we get $2-y^2+y^2-2y+1 = 1$, that is $y = 1$ and $x = \pm 1$. So the region is between $x = -1$ and $x = 1$, above $y = 1 - \sqrt{1-x^2}$ and below $y = \sqrt{2-x^2}$. Thus an integral representing the region is $\int_{-1}^{1} \sqrt{2-x^2} - (1-\sqrt{1-x^2})\,dx$. (The area is $\pi - 1$.) □

2.5. **Exercise 6.3.36.** Find volume of the solid obtained by rotating the region enclosed by the graphs $y = x^2$, $y = 12 - x$, and $x = 0$ about the axis $y = 15$.

Solution. Solving $x^2 = 12 - x$ gives $x = 3$ or -4. We'll take the $x \geq 0$ portion, since the $x \leq 0$ portion intersects the axis and is weird. (N.B., this problem is not very clearly written.) The distances are $15 - x^2$ and $15 - 12 + x = 3 + x$, respectively. Thus the volume is given by $V = \pi \int_0^3 (15-x^2)^2 - (3+x)^2\,dx = \pi \left[x^5/5 - 31x^3/3 - 3x^2 + 216x\right]_0^3 = 1953\pi/5$. □

2.6. **Exercise 6.3.54.** Verify the formula

$$\int_{x_1}^{x_2} (x-x_1)(x-x_2)\,dx = \frac{1}{6}(x_1-x_2)^3.$$

Then prove that the solid obtained by rotating the shaded region above $y = mx + c$ and below $y^2 = ax + b$ about the x-axis has volume $V = \frac{\pi}{6}BH^2$, with B the base of the region and H the height.

Solution. The verification is a trivial process, and is left as an exercise to the student. Let x_1 and x_2 be the roots of $f(x) = ax + b - (mx + c)^2$, where $x_1 < x_2$. Notice that $V = \pi \int_{x_1}^{x_2} f(x)\,dx$. Since x_1 and x_2 are roots, and the coefficient of x^2 is $-m^2$, we get that $f(x) = -m^2(x-x_1)(x-x_2)$. Thus the integral becomes $V = \pi \int_{x_1}^{x_2} (-m^2)(x-x_1)(x-x_2)\,dx$. By the formula given, we get $V = -m^2\pi \cdot \frac{1}{6}(x_1-x_2)^3$. Notice that $x_2 - x_1 = B$ and $m = \frac{H}{B}$, we finally get $V = \frac{\pi}{6}BH^2$, as desired. □