MATH 32A DISCUSSION

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1. Vectors

1.1. **Exercise 13.2.18.** Given $\mathbf{a} = 4\mathbf{i} + \mathbf{j}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j}$. Find $\mathbf{a} + \mathbf{b}$, $2\mathbf{a} + 3\mathbf{b}$, $|\mathbf{a}|$, and $|\mathbf{a} - \mathbf{b}|$.

Solution. $\mathbf{a} + \mathbf{b} = 5\mathbf{i} - \mathbf{j}$, $2\mathbf{a} + 3\mathbf{b} = 11\mathbf{i} - 4\mathbf{j}$, $|\mathbf{a}| = \sqrt{4^2 + 1^2}$, $\mathbf{a} - \mathbf{b} = 3\mathbf{i} + 3\mathbf{j}$, so $|\mathbf{a} - \mathbf{b}| = 3\sqrt{2}$. □

1.2. **Exercise 13.2.35.** Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

Solution. Tangent line has slope $y' = 2x$ with $x = 2$, so slope 4. Take $\langle 1, 4 \rangle$ and normalise to get $\left\langle 1/\sqrt{17}, 4/\sqrt{17} \right\rangle$. We also get the negative of that. □

1.3. **Exercise 13.2.45.** Use vectors to prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

Solution. Let $\overrightarrow{AB} = 2\mathbf{a}$, $\overrightarrow{BC} = 2\mathbf{b}$. Then the vector representing the midline is $\mathbf{a} + \mathbf{b}$ whereas the third side is $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = 2(\mathbf{a} + \mathbf{b})$. □

2. Dot Products

2.1. **Basics.** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ then the *dot product* is given by $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ then $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta$.

2.2. **Exercise 13.3.7.** Given $\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} + 9\mathbf{k}$. Find $\mathbf{a} \cdot \mathbf{b}$.

Solution. We have $\mathbf{a} \cdot \mathbf{b} = 1 \cdot 5 - 2 \cdot 0 + 3 \cdot 9 = 32$. □

2.3. **Exercise 13.3.56.** Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

Solution. Let $\mathbf{a}$ and $\mathbf{b}$ represent two sides, then $|\mathbf{a}| = |\mathbf{b}|$. The two diagonals are $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$. They are perpendicular if their dot product is zero. But $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 - |\mathbf{b}|^2 = 0$, as desired. □

2.4. **Exercise 13.3.60.** Show that if $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are orthogonal, then the vectors $\mathbf{a}$ and $\mathbf{b}$ must have the same length.

Solution. Notice $0 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = |\mathbf{a}|^2 - |\mathbf{b}|^2 = (|\mathbf{a}| + |\mathbf{b}|)(|\mathbf{a}| - |\mathbf{b}|)$. □

3. Cross Products

3.1. **Basics.** If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = b_1, b_2, b_3$ then the *cross product* is given by the mnemonic

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ then $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$, which is the area of the parallelogram determined by the vectors.

3.2. **Exercise 13.4.18.** If $\mathbf{a} = \langle 3, 1, 2 \rangle$, $\mathbf{b} = \langle -1, 1, 0 \rangle$, and $\mathbf{c} = \langle 0, 0, -4 \rangle$, show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

Solution. Notice that $\mathbf{a} \times \mathbf{b} = \langle -2, -2, 4 \rangle$, so $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \langle 8, -8, 0 \rangle$. But $\mathbf{b} \times \mathbf{c} = \langle -4, -4, 0 \rangle$, so $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \langle 8, -8, -8 \rangle$. □

3.3. **Exercise 13.4.30.** Let $P(2, 1, 5)$, $Q(-1, 3, 4)$, $R(3, 0, 6)$. Find a nonzero vector orthogonal to the plane through the points P, Q, and R, and find the area of triangle PQR.

Solution. Notice $\overrightarrow{PQ} = \langle -3, 2, -1 \rangle$ and $\overrightarrow{PR} = \langle 1, -1, 1 \rangle$, so $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1, 2, 1 \rangle$. This vector works. And the area of triangle is $\frac{1}{2}\,|\langle 1, 2, 1 \rangle| = \sqrt{6}/2$. □

3.4. **Exercise 13.4.48.** Prove that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{d} & \mathbf{b} \cdot \mathbf{d} \end{vmatrix}.$$

Solution. Use $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. Indeed, we get $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{a} \cdot (\mathbf{b} \times (\mathbf{c} \times \mathbf{d})) = \mathbf{a} \cdot ((\mathbf{b} \cdot \mathbf{d})\mathbf{c}) - (\mathbf{b} \cdot \mathbf{c})\mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$, as desired. □