MATH 32A DISCUSSION

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1. Calculus of Vector Functions

1.1. **Exercise 14.2.14.** Find the derivative of the vector function

$$\mathbf{r}(t) = \langle at\cos 3t, b\sin^3 t, c\cos^3 t\rangle.$$

Solution. Take the derivative component-wise:

$$\mathbf{r}'(t) = \langle a\cos 3t - 3at\sin 3t, 3b\sin^2 t\cos t, -3c\cos^2 t\sin t\rangle.$$

□

1.2. **Exercise 14.2.38.** Evaluate the integral $\int(\cos\pi t\,\mathbf{i} + \sin\pi t\,\mathbf{j} + t\,\mathbf{k})dt$.

Solution. Integrate component-wise: $\frac{1}{\pi}\sin\pi t\,\mathbf{i} - \frac{1}{\pi}\cos\pi t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k}$. □

1.3. **Exercise 14.2.45.** If $\mathbf{u}(t) = \langle\sin t, \cos t, t\rangle$ and $\mathbf{v}(t) = \langle t, \cos t, \sin t\rangle$, find

$$\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)].$$

Solution. Recall that $[\mathbf{u}(t)\cdot\mathbf{v}(t)]' = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$. Since $\mathbf{u}'(t) = \langle\cos t, -\sin t, 1\rangle$ and $\mathbf{v}'(t) = \langle 1, -\sin t, \cos t\rangle$, we get $t\cos t - \sin t\cos t + \sin t + \sin t - \sin t\cos t + t\cos t$. □

1.4. **Exercise 14.2.47.** Show that if $\mathbf{r}$ is a vector function such that $\mathbf{r}''$ exists, then

$$\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}(t)\times\mathbf{r}''(t).$$

Solution. Recall that $[\mathbf{u}(t)\times\mathbf{v}(t)]' = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$. So $[\mathbf{r}(t)\times\mathbf{r}'(t)]' = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\mathbf{r}''(t) = \mathbf{r}(t)\times\mathbf{r}''(t)$ since $\mathbf{u}(t)\times\mathbf{u}(t) = 0$ for any $\mathbf{u}$. □

1.5. **Basics of Differential Geometry.** Unit tangent vector is given by $\mathbf{T}(t) = \mathbf{r}'(t)/\left|\mathbf{r}'(t)\right|$. Arc length is $L = \int_a^b \left|\mathbf{r}'(t)\right| dt$. Curvature is $\kappa(t) = \left|\mathbf{T}'(t)\right|/\left|\mathbf{r}'(t)\right| = \left|\mathbf{r}'(t)\times\mathbf{r}''(t)\right|/\left|\mathbf{r}'(t)\right|^3 = \left|f''(x)\right|/[1+(f'(x))^2]^{3/2}$. Unit normal is given by $\mathbf{N}(t) = \mathbf{T}'(t)/\left|\mathbf{T}'(t)\right|$. The binormal is $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t)$.

1.6. **Exercise 14.3.4.** Find the length of the curve $\mathbf{r}(t) = \langle\cos t, \sin t, \ln\cos t\rangle$, $0 \le t \le \pi/4$.

Solution. First calculate $\left|\mathbf{r}'(t)\right| = \left|\langle -\sin t, \cos t, -\sin t/\cos t\rangle\right| = \sec t$. The arc length is $L = \int_0^{\pi/4}\sec t\,dt$. □

1.7. **Exercise 14.3.11.** Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the exact length of C from the origin to the point $(6, 18, 36)$.

Solution. The curve is $\mathbf{r}(t) = \langle \sqrt{2t}, t, t\sqrt{2t}/3 \rangle$. Then $\mathbf{r}'(t) = \left\langle 1/\sqrt{2t}, 1, \sqrt{t/2} \right\rangle$. Now $|\mathbf{r}'(t)| = \sqrt{1/2t + 1 + t/2} = (\sqrt{t} + 1/\sqrt{t})/\sqrt{2}$. So arc length is $L = \int_0^1 8(\sqrt{t} + 1/\sqrt{t})/\sqrt{2}\, dt = (\frac{2}{3}t^{3/2} + 2\sqrt{t})/\sqrt{2}|_0^{18} = 42$.

□

1.8. **Exercise 14.3.17.** Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$, and find the curvature of the curve $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t \rangle$.

Solution. Simply calculate: $\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t \rangle$, $|\mathbf{r}'(t)| = \sqrt{29}$, $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)| = \frac{1}{\sqrt{29}}\langle 2\cos t, 5, -2\sin t \rangle$; $\mathbf{T}'(t) = \frac{1}{\sqrt{29}}\langle -2\sin t, 0, -2\cos t \rangle$, $|\mathbf{T}'(t)| = \frac{2}{\sqrt{29}}$, $\mathbf{N}(t) = \mathbf{T}'(t)/|\mathbf{T}'(t)| = \langle -\sin t, 0, -\cos t \rangle$; $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)| = \frac{2}{29}$. □

1.9. **Exercise 14.3.23.** Find the curvature of $\mathbf{r}(t) = \langle 3t, 4\sin t, 4\cos t \rangle$.

Solution. We have $\mathbf{r}'(t) = \langle 3, 4\cos t, -4\sin t \rangle$, $\mathbf{r}''(t) = \langle 0, -4\sin t, -4\cos t \rangle$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle -16, 12\cos t, -12\sin t \rangle$, $|\mathbf{r}'(t)| = 5$. So $\kappa(t) = |\mathbf{r}'(t) \times \mathbf{r}''(t)|/|\mathbf{r}'(t)|^3 = 20/5^3 = 4/25$.

Alternatively, $\mathbf{T}(t) = \mathbf{r}'(t)/|\mathbf{r}'(t)| = \frac{1}{5}\langle 3, 4\cos t, -4\sin t \rangle$, $\mathbf{T}'(t) = \mathbf{r}''(t)/|\mathbf{r}'(t)| = \frac{1}{5}\langle 0, -4\sin t, -4\cos t \rangle$, $|\mathbf{T}'(t)| = |r''(t)|/|\mathbf{r}'(t)|^2 = \frac{4}{5}$, and $\kappa(t) = |\mathbf{T}'(t)|/|\mathbf{r}'(t)| = \frac{4}{5}/5 = 4/25$. □

1.10. **Exercise 14.3.44.** Given $\mathbf{r}(t) = \langle \cos t, \sin t, \ln\cos t \rangle$. Find the vectors $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at point $(1, 0, 0)$.

Solution. First calculate $|\mathbf{r}'(t)| = |\langle -\sin t, \cos t, -\sin t/\cos t \rangle| = \sec t$. □