

## MATH 32A DISCUSSION

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## 1. HIGHER PARTIAL DERIVATIVES

1.1. **Clairaut's Theorem.** If  $f_{xy}$  and  $f_{yx}$  are continuous then  $f_{xy} = f_{yx}$ .

1.2. **Exercise 15.3.60.** Verify that the conclusion Clairaut's Theorem holds for  $u = xye^y$ . That is,  $u_{xy} = u_{yx}$ .

*Solution.* Notice  $u_x = ye^y$ ,  $u_{xy} = e^y + ye^y$ , and  $u_y = xe^y + xye^y$ ,  $u_{yx} = e^y + ye^y$ .  $\square$

1.3. **Exercise 15.3.82.** For an ideal gas of mass  $m$  at absolute temperature  $T$ , pressure  $P$ , and volume  $V$ , we have  $PV = mRT$ , where  $R$  is the gas constant. Show that  $\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1$ .

*Solution.* Notice  $\frac{\partial P}{\partial V} = -mRT/V^2$ ,  $\frac{\partial V}{\partial T} = mR/P$ ,  $\frac{\partial T}{\partial P} = V/mR$ , so multiplying gives  $-1$ , recognising that  $mRT = PV$ .  $\square$

1.4. **Exercise 15.3.87.** Is it possible that there is a function  $f$  whose partial derivatives are  $f_x(x, y) = x + 4y$  and  $f_y(x, y) = 3x - y$ ?

*Solution.* Negative, since  $f_{xy} = 4$  and  $f_{yx} = 3$  are continuous but not equal, contradicting Clairaut's theorem.  $\square$

1.5. **Exercise 15.3.91.** Use Clairaut's Theorem to show that if the third-order partial derivatives of  $f$  are continuous, then  $f_{xxy} = f_{yxy} = f_{yyx}$ .

*Solution.* Clairaut gives  $f_{xy} = f_{yx}$ , taking the partial with respect to  $y$  gives  $f_{xyy} = f_{yxxy}$ . Applying Clairaut to  $f_y$  gives  $f_{yxy} = f_{yyx}$ .  $\square$

1.6. **Exercise 15.3.92.** How many  $k$ th-order partial derivatives does a function of two variables have? How about  $n$  variables? If these partial derivatives are all continuous, how many of them can be distinct?

*Solution.* Counting mixed partials as different, we get  $2^k$  or  $n^k$  for 2 or  $n$  variables, respectively. If they are all continuous, then for the case of 2 variables, we only care how many times we take derivative with respect to  $x$ , which could be any integer times between 0 and  $k$ , inclusive. So there are  $k + 1$  possibly different partial derivatives. As for general  $n$ , this requires knowing partitions of integers, and is beyond the scope of this class. [Talk to me if you are interested; this is an area known as combinatorics, which is my main field of focus.]  $\square$

## 2. TANGENT PLANES AND LINEAR APPROXIMATIONS

**2.1. Basics.** Assume  $f$  has continuous partial derivatives. An equation of the tangent plane to the surface  $z = f(x, y)$  at the point  $P(x_0, y_0, z_0)$  is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

If the partial derivatives  $f_x$  and  $f_y$  exist near  $(a, b)$  and are continuous at  $(a, b)$ , then  $f$  is differentiable at  $(a, b)$ .

If  $z = f(x, y)$ , then the *total differential* is given by

$$dz = f_x(x, y) dx + f_y(x, y) dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

**2.2. Exercise 15.4.16.** Explain why  $f(x, y) = \sin(2x + 3y)$  is differentiable at  $(-3, 2)$  and find the linearisation at that point.

*Solution.* Notice that  $f_x(x, y) = 2 \cos(2x + 3y)$  and  $f_y(x, y) = 3 \cos(2x + 3y)$  are continuous, hence we obtain differentiability. The linearisation is  $z - f(-3, 2) = f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2)$ , which is  $z = 2(x + 3) + 3(y - 2)$ .  $\square$

**2.3. Exercise 15.4.33.** The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.

*Solution.* Let  $\ell = 30$ ,  $w = 24$ , and  $A = w\ell$ ,  $d\ell = dw = 0.1$ . Then  $dA = \frac{\partial A}{\partial \ell} d\ell + \frac{\partial A}{\partial w} dw = w d\ell + \ell dw = 5.4 \text{ cm}^2$ .  $\square$

**2.4. Exercise 15.4.40.** Four positive numbers, each less than 50, are rounded to the first decimal place and then multiplied together. Use differentials to estimate the maximum possible error in the computed product that might result from the rounding.

*Solution.* Similar to above, we get  $4 \cdot 50^3 \cdot 0.5$ .  $\square$

**2.5. Exercise 15.4.42.** Find an equation of the tangent plane to a surface  $S$  at point  $P(2, 1, 3)$ , given that the curves  $\mathbf{r}_1(t) = \langle 2 + 3t, 1 - t^3, 3 - 4t + t^2 \rangle$  and  $\mathbf{r}_2(u) = \langle 1 + u^2, 2u^3 - 1, 2u + 1 \rangle$  both lie on  $S$ .

*Solution.* The cross product of the tangent vectors of the curves at the point specified gives a normal.  $\square$

**2.6. Exercise 15.4.45.** Prove that if  $f$  is a function of two variables that is differentiable at  $(a, b)$ , then  $f$  is continuous at  $(a, b)$ .

*Solution.* Recall that there exists  $\varepsilon_1, \varepsilon_2$  that approaches 0 as  $(\Delta x, \Delta y) \rightarrow (0, 0)$  such that  $f(a + \Delta x, b + \Delta y) - f(a, b) = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$ . Taking the limit  $(\Delta x, \Delta y) \rightarrow (0, 0)$  of both sides, the right side becomes zero. So  $\lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} f(a + \Delta x, b + \Delta y) = f(a, b)$ , as desired.  $\square$