MATH 32A DISCUSSION

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1. Chain Rule

1.1. **Exercise 15.5.1. Chain rule case 1.** Given $z = x^2 + y^2 + xy$, $x = \sin t$, $y = e^t$, find dz/dt.

Solution. Recall chain rule case 1: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentaible functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Here $\frac{\partial z}{\partial x} = 2x + y$, $\frac{\partial z}{\partial y} = 2y + x$, $\frac{dx}{dt} = \cos t$, $\frac{dy}{dt} = e^t$. □

1.2. **Exercise 15.5.10. Chain rule case 2.** Given $z = e^{x+2y}$, $x = s/t$, $y = t/s$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution. Recall chain rule case 2: Suppose $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$ are differentiable, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial z}{\partial s},$$

and similarly for t.

So $\frac{\partial z}{\partial s} = e^{x+2y}(\frac{1}{t} - \frac{2t}{s^2})$, and $\frac{\partial z}{\partial t} = e^{x+2y}(\frac{2}{s} - \frac{s}{t^2})$. □

1.3. **Exercise 15.5.26. Chain rule general version.** Given $Y = w \arctan(uv)$, $u = r + s$, $v = s + t$, $w = t + r$. Find $\frac{\partial Y}{\partial r}$, $\frac{\partial Y}{\partial s}$, $\frac{\partial Y}{\partial t}$ when $r = 1$, $s = 0$, and $t = 1$.

Solution. Recall chain rule general version: Suppose that u is a differentiable function of the n variables $x_1, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, \ldots, t_m$. Then u is a function of $t_1, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i},$$

for each $i = 1, \ldots, m$.

First calculate $u = 1$, $v = 1$, $w = 2$. So at these points, we get $\frac{\partial Y}{\partial w} = \arctan(uv) = \pi/4$, $\frac{\partial Y}{\partial u} = \frac{wv}{1+(uv)^2} = 1$, and $\frac{\partial Y}{\partial v} = 1$ by symmetry. Now $\frac{\partial Y}{\partial r} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial r} = \frac{\partial Y}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial Y}{\partial w}\frac{\partial w}{\partial r} = \frac{\partial Y}{\partial u} + \frac{\partial Y}{\partial w} = 1 + \pi/4$. Similarly, $\frac{\partial Y}{\partial s} = \frac{\partial Y}{\partial u} + \frac{\partial Y}{\partial v} = 2$, and $\frac{\partial Y}{\partial t} = 1 + \pi/4$. □

1.4. **Exercise 15.5.29. Implicit differentiation.** Given $\cos(x - y) = xe^y$, find dy/dx.

Solution. Recall implicit differentiation: Suppose $F(x,y)=0$ defines y implicitly as a differentiable function of x, i.e., $y=f(x)$ where $F(x,f(x))=0$ for all x in the domain of f. Then if F is differentiable, $F_y \neq 0$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

So let $F(x,y)=\cos(x-y)-xe^y$. Then $F_x = -\sin(x-y)-e^y$, $F_y = \sin(x-y)-xe^y$, hence $\frac{dy}{dx} = \frac{\sin(x-y)+e^y}{\sin(x-y)+xe^y}$. □

1.5. **Exercise 15.5.43.** One side of a triangle is increasing at a rate of 3 cm/s and a second side is decreasing at a rate of 2 cm/s. If the area of the triangle remains constant, at what rate does the angle between the sides change when the first side is 20 cm long, the second side is 30 cm, and the angle is $\pi/6$.

Solution. Let the triangle have sides $a=20$, $b=30$, with angle between them $C=\pi/6$. We know that $\frac{da}{dt}=3$ and $\frac{db}{dt}=-2$. The area A of the triangle is $A=\frac{1}{2}ab\cos C$. Taking the derivative of $2A$ with respect to t, we get $2\frac{dA}{dt} = b\cos C\frac{da}{dt} + a\cos C\frac{db}{dt} - ab\sin C\frac{dC}{dt}$. Now $\frac{dA}{dt}=0$, so we may solve for $\frac{dC}{dt} = -\frac{1}{12\sqrt{3}}$. □

1.6. **Exercise 15.5.49.** Assume all given functions have continuous second-order partial derivatives. Show that any function of the form $z=f(x+at)+g(x-at)$ is a solution of the wave equation $\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}$.

Solution. Let $u=x+at$, $v=x-at$. Then $z=f(u)+g(v)$. Notice $\frac{\partial z}{\partial t} = af'(u)-ag'(v)$, $\frac{\partial^2 z}{\partial t^2} = a^2f''(u)+a^2g''(v)$, whereas $\frac{\partial z}{\partial x} = f'(u)+g'(v)$, $\frac{\partial^2 z}{\partial x^2} = f''(u)+g''(v)$. □

1.7. **Exercise 15.5.53.** Assume all given functions have continuous second-order partial derivatives. If $z=f(x,y)$, where $x=r\cos\theta$ and $y=r\sin\theta$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}.$$

Solution. This follows from straight-forward, albeit tedious, calculations. We get $\frac{\partial z}{\partial r} = f_x\cos\theta + f_y\sin\theta$. Now when calculating $\frac{\partial^2 z}{\partial r^2}$, we get $\frac{\partial^2 z}{\partial r^2} = f_{xr}\cos\theta + f_{yr}\sin\theta$, where $f_{xr} = f_{xx}\cos\theta + f_{xy}\sin\theta$ and similarly for f_{yr}. Similarly, $\frac{\partial z}{\partial \theta} = -f_x r\sin\theta + f_y r\cos\theta$, so, using the product rule, we get $\frac{\partial^2 z}{\partial \theta^2} = -f_{x\theta}r\sin\theta - f_x r\cos\theta + f_{y\theta}r\cos\theta - f_y r\sin\theta$, where $f_{x\theta} = -f_{xx}r\sin\theta + f_{xy}r\cos\theta$, and similarly for $f_{y\theta}$. Putting this all together, we see the desired result. □

1.8. **Exercise 15.5.58.** Suppose that the equation $F(x,y,z)=0$ implicitly defines each of the three variables x, y, and z as functions of the other two: $z=f(x,y)$, $y=g(x,z)$, and $x=h(y,z)$. If F is differentiable and F_x, F_y, and F_z are all nonzero, show that

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1.$$

Solution. First consider z as a function of x and y, treat y as a constant, and implicitly differentiate $F(x,y,z)=0$ to find $\frac{\partial z}{\partial x}$. Notice $\frac{\partial F}{\partial x} = F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x}$, where $\frac{\partial x}{\partial x}=1$, $\frac{\partial y}{\partial x}=0$ since y is held constant. So we get $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$. By symmetry, we get $\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}$, $\frac{\partial y}{\partial z} = -\frac{F_z}{F_y}$. Multiplying, we get the desired result. □