Math 1271-040 Midterm Exam 1 Solutions

Problem 1. Let $f(x) = x^3 + 3x^2 + 5$.

(a) Calculate $f''(x)$.

Solution. $f'(x) = 3x^2 + 6x$; $f''(x) = 6x + 6$. □

(b) Suppose c is a number such that $f''(c) = 0$. Determine the value of c.

Solution. $f''(c) = 6c + 6 = 0$ implies $c = -1$. □

(c) Find an equation of the tangent line to the graph of $f(x)$ at $x = c$, where c is the value determined in part (b).

Solution. $f(c) = f(-1) = -1 + 3 + 5 = 7$ and $f'(c) = f'(-1) = 3 - 6 = -3$ yields an equation

$$y - 7 = -3(x + 1)$$

for the tangent line. □

Problem 2. Evaluate the limits. Simplify answers but leave them exact (*e.g.*, do not use decimal approximations). Answers could be ∞, $-\infty$, or "does not exist."

(a) $\lim_{\theta \to \pi} \theta^2 + \cos\theta$

Solution. As $\theta^2 + \cos\theta$ is continuous, the Direct Substitution Property yields

$$\pi^2 + \cos(\pi) = \pi^2 - 1.$$

□

(b) $\lim_{x \to 2} \frac{e^{5x} - e^{10}}{x - 2}$

Solution. We recognize the limit as $f'(2)$ for $f(x) = e^{5x}$. As $f'(x) = 5e^{5x}$, we get

$$f'(2) = 5e^{10}.$$

□

(c) $\lim_{x \to -\infty} x + \sqrt{x^2 - 3x}$

Solution. Multiplying by the conjugate yields

$$\lim_{x \to -\infty} \frac{x^2 - (x^2 - 3x)}{x - \sqrt{x^2 - 3x}} = \lim_{x \to -\infty} \frac{3x}{x - \sqrt{x^2 - 3x}}.$$

Dividing both the numerator and the denominator by x while remembering that $x = -\sqrt{x^2}$ for $x < 0$ yields

$$\lim_{x \to -\infty} \frac{\frac{3x}{x}}{\frac{x}{x} - \frac{1}{-\sqrt{x^2}}\sqrt{x^2 - 3x}} = \lim_{x \to -\infty} \frac{3}{1 + \sqrt{1 - \frac{3}{x}}} = \frac{3}{1 + \sqrt{1 - 0}} = \frac{3}{2}.$$

□

Problem 3. Differentiate. It is not necessary to simplify answers.

(a) $f(x) = (5x-7)^2(2x^{23}-x)^3$

Solution. $f'(x) = 2(5x-7)(5)(2x^{23}-x)^3 + (5x-7)^2(3)(2x^{23}-x)^2(46x^{22}-1)$ □

(b) $f(x) = e^{\sqrt{x-e^{3x}}}$

Solution. $f'(x) = e^{\sqrt{x-e^{3x}}} \cdot \frac{1}{2}(x-e^{3x})^{-1/2} \cdot (1-3e^{3x})$ □

(c) $f(x) = \dfrac{3x^8+2x-7}{\sqrt[3]{x}}$

Solution. Differentiate $f(x) = 3x^{23/3} + 2x^{2/3} - 7x^{-1/3}$ to get

$$f'(x) = 23x^{20/3} + \tfrac{4}{3}x^{-1/3} + \tfrac{7}{3}x^{-4/3}.$$

□

Problem 4.

(a) Write down the definition of the derivative of a function f at a point a.

Solution. $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}.$ □

(b) Find the derivative of $f(x) = \dfrac{1}{\sqrt{5x}}$ using the definition of the derivative. Do not use differentiation rules.

Solution. By definition, we have

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{5(x+h)}} - \frac{1}{\sqrt{5x}}\right) = \lim_{h\to 0} \frac{\sqrt{5x}-\sqrt{5(x+h)}}{h\sqrt{5(x+h)}\sqrt{5x}}.$$

Multiply by the conjugate to get

$$\lim_{h\to 0} \frac{5x-5(x+h)}{h\sqrt{5(x+h)}\sqrt{5x}(\sqrt{5x}+\sqrt{5(x+h)})} = \lim_{h\to 0} \frac{-5h}{h\sqrt{5(x+h)}\sqrt{5x}(\sqrt{5x}+\sqrt{5(x+h)})}.$$

Cancelling the h and direct substitution yields

$$\lim_{h\to 0} \frac{-5}{\sqrt{5(x+h)}\sqrt{5x}(\sqrt{5x}+\sqrt{5(x+h)})} = \frac{-5}{\sqrt{5x}\sqrt{5x}(\sqrt{5x}+\sqrt{5x})} = \frac{-1}{2\sqrt{5}x^{3/2}},$$

as desired. □

Problem 5. Prove the following statements. Justify answers and cite theorems used.

(a) $\lim_{t \to 0} t^3(t + \cos \frac{1}{t^2}) = 0$

Solution. As

$$-1 \leq \cos \tfrac{1}{t^2} \leq 1,$$

we have

$$-\left|t^3\right| \leq t^3 \cos \tfrac{1}{t^2} \leq \left|t^3\right|.$$

Since $\lim_{t \to 0} -\left|t^3\right| = \lim_{t \to 0} \left|t^3\right| = 0$, the Squeeze Theorem says

$$\lim_{t \to 0} t^3 \cos \tfrac{1}{t^2} = 0.$$

Therefore

$$\lim_{t \to 0} t^3(t + \cos \tfrac{1}{t^2}) = \lim_{t \to 0} t^4 + \lim_{t \to 0} t^3 \cos \tfrac{1}{t^2} = 0 + 0 = 0,$$

as desired. □

Grading. 1pt for correctly identifying the bounds of $\cos \frac{1}{t^2}$; 1pt for getting the correct form before applying the Squeeze Theorem; 1pt for writing down the use of Squeeze Theorem.

(b) The function $f(x) = \sin x + \frac{\pi}{x}$ has a root in the interval $(-10\pi, 10\pi)$.

Solution. As $\sin x$ is continuous everywhere and $\frac{\pi}{x}$ is continuous on its domain, $f(x)$ is continuous everywhere except at $x = 0$. In particular, it is continuous on the subinterval $(\pi/2, 3\pi/2)$. Note that $f(\pi/2) = 1 + 2 = 3$ and $f(3\pi/2) = -1 + \frac{2}{3} = -\frac{1}{3}$. By the Intermediate Value Theorem, as f is continuous on $(\pi/2, 3\pi/2)$ and $f(\pi/2) > 0 > f(3\pi/2)$, there is a root in the interval, as desired. □

Grading. 1pt for showing the function is continuous except at $x = 0$; 1pt for picking an interval where the outputs change signs; 1pt for citing the Intermediate Value Theorem.

Problem	Mean	Stdev
Problem 1 (6 points)	4.17	1.70
Problem 2 (6 points)	2.07	0.80
Problem 3 (6 points)	4.31	1.41
Problem 4 (6 points)	3.26	1.49
Problem 5 (6 points)	2.43	1.69
$\sum$ (30 points total)	16.24	5.04