Math 4707 Catalan Numbers

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Enumerative combinatorics is a field of mathematics that deals with counting the number of objects satisfying some combinatorial description. Catalan numbers count a wide variety of objects.

Problem 1. A *monotonic path* is a finite sequence of 'up' and 'right' steps of unit length. Count the number of monotonic paths from $(0,0)$ to (n,m).

Problem 2. Consider monotonic paths from $(0,0)$ to (n,n) which cross above the diagonal, i.e., those that enter the region $y > x$. Count the number of such paths by establishing a bijection with monotonic paths from $(0,0)$ to $(n-1,n+1)$.

Problem 3. Count the number of monotonic paths from $(0,0)$ to (n,n) which do not cross above the diagonal, i.e., those that stay in the region $y \le x$.

Problem 4. Write the answer from the previous exercise as a fraction of a binomial coefficient. This is known as the *Catalan number* C_n.

Problem 5. Let $\nearrow = (1,1)$ and $\searrow = (1,-1)$ be vectors in $\mathbb{Z}^2$. Let

$$S = \left\{ (v_1, v_2, \ldots, v_{2n}) \in \{\nearrow, \searrow\}^{2n} : \sum_{i=1}^{2n} v_i = (2n, 0) \right\}$$

be the collection of all sequences of length $2n$ with the same number of $\nearrow$ as $\searrow$. Count the number $|S|$ of such sequences.

Problem 6. Let B be the sequences $(v_1, \ldots, v_{2n}) \in S$ such that each partial sum $s_t = \sum_{i=1}^{t} v_i$ has non-negative y-coordinate for $t \in [2n] = \{1, 2, \ldots, 2n\}$. These are known as *ballot sequences*—a way for $2n$ people to cast yes/no votes one at a time so that there are always at least as many "yes" as "no" votes, with a tie as the end result. Count the number $|B|$ of ballot sequences.

Problem 7. Prepend $v_0 = \nearrow$ in front of every sequence in both S and B, and call the resulting sequences *augmented* and denote the sets $\overline{S}$ and $\overline{B}$, respectively. For each augmented sequence $(v_0, v_1, \ldots, v_{2n}) \in \overline{S}$, show that there is a unique $r \in [0, 2n] = \{0, 1, \ldots, 2n\}$ such that $(v_{0+r}, v_{1+r}, \ldots, v_{2n+r})$ is an augmented ballot sequence in $\overline{B}$, where the indices are read modulo $2n+1$. Call the map $f : \overline{S} \to \overline{B}$ that sends the augmented sequence to its corresponding augmented ballot sequence.

Problem 8. Show that f is surjective and that the fibers under f are all of the same size, and calculate this size. That is, calculate $\left|f^{-1}(b)\right|$ for $b \in B$.

Problem 9. Establish a linear relationship between $|B|$ and $|S|$.

Problem 10. Conclude a second proof of the expression of the Catalan number in **Problem 4** *without* using subtraction, but using division instead.