Math 4707
Chromatic Polynomials

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The *chromatic polynomial* of a simple graph G, $C_G(\lambda)$, is the number of ways of properly coloring the vertices of G using λ colors. For example, if G is the complete graph K_n, then there are λ ways of coloring one vertex, $\lambda - 1$ ways to color another, etc. Therefore, $C_{K_n}(\lambda) = \lambda(\lambda-1)\ldots(\lambda-n+1)$.

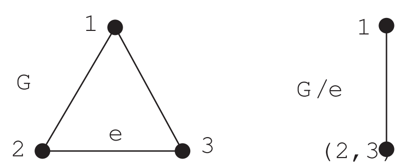
As another example, if G is the null graph N_n (n vertices and no edges), then there are λ ways to color each vertex, so $C_{N_n}(\lambda) = \lambda^n$.

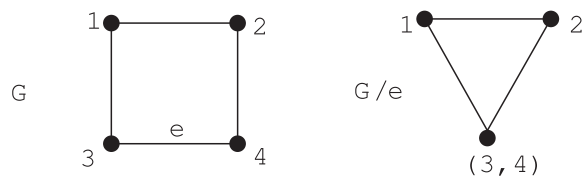
As a third example, let T be any tree with n vertices.

Theorem 1. *If T is a tree, then $C_T(\lambda) = \lambda(\lambda - 1)^{n-1}$.*

Proof. By induction on the number of vertices of T. It is clearly true when T has 2 vertices (it is K_2). Now suppose it is true for any tree with $n-1$ vertices and let T be a tree with n vertices. Pick a terminal vertex, v. Prune v from T, leaving T', which has $n-1$ vertices. By induction, there are $\lambda(\lambda-1)^{n-2}$ ways to color T' using no more than λ colors. But then there are $\lambda - 1$ ways to color v (since it cannot be colored the same as the vertex to which it is adjacent), giving $\lambda(\lambda-1)^{n-1}$ ways to color T using no more than λ colors. □

An important tool for computing chromatic polynomials is the *deletion-contraction recursion.* Let e be an edge in the simple graph G. Let $G - e$ be the graph obtained by removing e and let G/e be the graph obtained by shrinking e to a single vertex (called "contracting"), or, equivalently, by removing e and pasting the two endpoints together (and removing multiple edges). Here are a couple of examples of contractions.





Theorem 2. *The chromatic polynomial of G satisfies the recursion*

$$C_G(\lambda) = C_{G-e}(\lambda) - C_{G/e}(\lambda)\,.$$

Proof. We prove instead

$$C_G(\lambda) + C_{G/e}(\lambda) = C_{G-e}(\lambda)\,.$$

Suppose e is incident upon u and v. Note that G is formed from $G-e$ by adding an edge between non-adjacent u and v and G/e is obtained from $G-e$ by pasting together non-adjacent vertices u and v. Now suppose f is a proper coloring of $G-e$. Either u and v are colored differently or are colored the same. In the former case, f will be a proper coloring of G, while in the latter case, f will be a proper coloring of G/e by coloring the combined u-v vertex the color of u. On the other hand, suppose f is a proper coloring of G. Then it is obviously a proper coloring of $G-e$ with u and v colored differently. Finally, if f is a proper coloring of G/e, then it is a proper coloring of $G-e$ by coloring both u and v the color of the combined u-v vertex. □

A great deal of information is contained in the chromatic polynomial. This theorem tells some of it.

Theorem 3. *Suppose G is a simple graph. Then $C_G(\lambda)$ is a polynomial in λ with the following properties.*

1. *The degree of $C_G(\lambda)$ is n, the number of vertices of G, and the coefficient of λ^n is 1.*
2. *The coefficient of λ^{n-1} is $-m$ where m is the number of edges of G.*
3. *The number of components of G is the lowest degree term with non-zero coefficient.*
4. *The signs of the coefficients of $C_G(\lambda)$ alternate.*

Proof. The proof is by induction on the number of edges, using the deletion-contraction theorem. The theorem is clearly true for the null graph (no edges), since $C_{N_n}(\lambda) = \lambda^n$.

Now suppose the theorem is true for all graphs with fewer than m edges and let G be a graph with m edges, $m \geq 1$. Pick an edge e and write

$$C_G(\lambda) = C_{G-e}(\lambda) - C_{G/e}(\lambda)\,.$$

Since, by induction, $C_{G-e}(\lambda)$ and $C_{G/e}(\lambda)$ are both polynomials in λ, $C_G(\lambda)$ will be a polynomial in λ. Also by induction, the degree of $C_{G-e}(\lambda)$ is n (with coefficient 1), while the degree of $C_{G/e}(\lambda)$ is $n-1$ (with coefficient 1). Therefore the degree of $C_G(\lambda)$ is n with coefficient 1.

By induction, the coefficient of λ^{n-1} in $C_{G-e}(\lambda)$ is the negative of the number of edges in $G - e$ and the coefficient of λ^{n-1} in $C_{G/e}(\lambda)$ is 1, so the coefficient of λ^{n-1} in $C_G(\lambda)$ is the negative of the number of edges in G.

By induction, the signs of $C_{G-e}(\lambda)$ alternate, starting with $+$ for λ^n, while the signs of $C_{G/e}(\lambda)$ alternate, starting with $+$ for λ^{n-1}. Therefore the signs of $C_G(\lambda)$ will alternate, starting with $+$ for λ^n.

Finally, by induction, the lowest power with non-zero coefficient of $C_{G-e}(\lambda)$ is the number of components of $G - e$, while the lowest power with non-zero coefficient of $C_{G/e}(\lambda)$ is the number of components of G/e. Since G/e has the same number of components as G and $G - e$ has either the same number or one more component as G, it follows that the lowest power with non-zero coefficient of $C_G(\lambda)$ is the number of components of G. $\square$

Note that the induction in this last proof was "strong induction." That is, the inductive hypothesis was that the statement was true for all graphs with fewer than m edges (not just $m - 1$ edges). We needed this stronger hypothesis because G/e might have $< m - 1$ edges.