Math 4707 Midterm 3

You may use books, notes, and calculators on this exam. Calculators will not be necessary. Please refrain from using other electronic devices such as laptops or cell phones. Do your work individually without collaboration. You have the full class period for the exam, and may leave early after turning in your work.

Assume that all graphs are simple, unless otherwise stated.

There are 5 problems, each worth 7 points. Please write your solutions carefully, showing all your work and justifying your steps rigorously. If you use tools from Chapter 2 such as induction, inclusion-exclusion, or pigeonholes, please state so explicitly.

Problem 1. Let P be a convex polyhedron in which every face is either a triangle or square. Prove that if every vertex has degree 4, then P has exactly 8 triangles.

Solution. Let P be a convex polyhedron with only triangle and square faces. Let v, e, t, and s denote the number of vertices, edges, triangle faces, and square faces, respectively. By Euler's formula, we have

$$v - e + t + s = 2.$$

By counting the number of incident pairs of edges and faces in two different ways, we get

$$2e = 3t + 4s.$$

Since every vertex is incident to four edges, and every edge is incident to two vertices, we get

$$4v = 2e.$$

Putting these together we get

$$8 = 4v - 4e + 4t + 4s = -(3t + 4s) + 4t + 4s = t.$$

We conclude that there are exactly 8 triangles, as desired. □

Problem 2. Prove that there is a way to colour the edges of $K_{n,m}$ red or blue such that there are at most

$$\binom{n}{a}\binom{m}{b}2^{1-ab}$$

monochromatic $K_{a,b}$ subgraphs.

Solution. Colour each edge randomly and independently red or blue with probability 1/2 each. For a choice $S \subseteq V(K_{n,m})$ consisting of a vertices on the left and b vertices on the right, the probability that S induces a $K_{a,b}$ is 2^{1-ab}, since there are 2 colours and ab edges. Therefore, by linearity of expectation, the expected number of monochromatic $K_{a,b}$ is $\binom{n}{a}\binom{m}{b}2^{1-ab}$. There is some colouring where the number is at most the expected number. □

Problem 3. Let G be a graph with girth $g(G) = 5$ (*i.e.*, containing no cycles of lengths 3 or 4). Prove that if G is d-regular (*i.e.*, every vertex has degree d), then G has at least d^2 vertices.

Solution. Fix a vertex $v \in G$, and consider its neighbours $u_1, u_2, \ldots, u_d$. Note that none of the u_i are neighbours of each other, lest there be a 3-cycle. Moreover, besides v, the neighbourhoods of u_i are otherwise disjoint, lest there be a 4-cycle. Each u_i has $d-1$ neighbours besides v, thus we have accounted for $1 + d + d(d-1) \geq d^2$ distinct vertices. □

Problem 4. Prove that a graph with m edges has a **bipartite** subgraph with at least $m/2$ edges.
[Hint: Divide the vertices into two parts randomly.]

Solution. Let G be a graph with m edges. For each vertex, randomly and independently assign it to either sets A or B with equal probability. Say an edge e is *crossing* if it has one endpoint in A and one in B. Note that the probability that e is crossing is exactly $1/2$, since the second vertex must be assigned a different set than the first vertex. Let X_e be the indicator that e is crossing, and $X = \sum_{e\in E(G)} X_e$ be the number of crossing edges. By linearity of expectation, we get

$$\mathbb{E}[X] = \sum_{e\in E(G)} \mathbb{E}[X_e] = m/2.$$

This means that there is a partition of vertices into sets A and B such that at least $m/2$ edges are crossing. Delete the non-crossing edges to obtain the desired bipartite subgraph. □

Problem 5. Your friend owns an art gallery P, which has the shape of an n-gon (a polygon with n sides, $n \geq 3$). He wants to place security cameras along its walls to guard the gallery. Consider each camera as a point on the boundary of P. A collection of cameras *guard* the polygon P if for every point p in the interior face of P, there is a camera c such that the straight line segment from c to p stays in P. For example, a single camera is sufficient to guard a *convex* polygon. He asks you for the minimum number of cameras he needs. You set out to show that, without knowing the shape of P, $\lfloor n/3 \rfloor$ cameras is always enough, and sometimes necessary. (Recall that $\lfloor x \rfloor$ denotes the largest integer n such that $n \leq x$; for simplicity, you may assume that n is divisible by 3.)

(a) Prove that any n-gon P can be **triangulated** without adding more vertices. More specifically, think of P as a cycle of length n, drawn in the plane with *straight* edges. Add *straight* edges between existing vertices so that each *interior* face is a triangle. The exterior face should be left untouched.

(b) The resulting graph has n vertices, all on the boundary, and each interior face is a triangle. Calculate its **chromatic number**, *i.e.*, the minimum number of colours needed to properly colour the graph.

(c) Show that $\lfloor n/3 \rfloor$ cameras is **sufficient** to guard P by placing at most that many cameras on the *vertices* of P. [Hint: Having triangulated P, your task is therefore to guard every single triangle.]

(d) Prove that $\lfloor n/3 \rfloor$ cameras is **sometimes needed** for *every* $n \geq 3$ by giving an *infinite family* of examples (one n-gon for each n) where that many cameras is required, *i.e.*, your example should work for an arbitrary value of n.

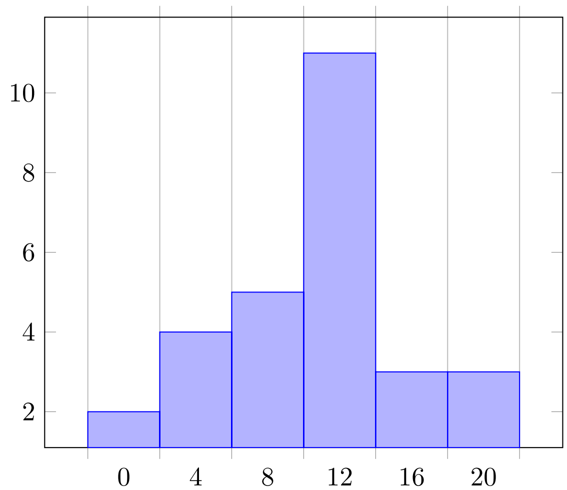
Solution. (a). The interior angles average to $\frac{n-2}{n}\pi$, so there must be some vertex whose interior angle is less than π. Pick such a vertex v and consider its two neighbours x and y. If xy is already an edge, then $n = 3$ and we are done. Otherwise, if we can add xy as a straight edge, we can then remove v and proceed by induction. Finally, if xy cannot be added, that means there is some vertex inside the triangle with v, x, y as vertices. This means v must have line-of-sight to some vertex z (consider a ray from v to x and sweet it towards y, stop when the ray hits a vertex). Join vz by a straight edge to divide P into two polygons; inductively triangulate each.

(b). Let T be a triangulation of P. Since there are triangles, it is obvious that we need at least 3 colours. We prove that $\chi(T) = 3$ colours is enough by induction. Consider the dual graph T^* of T and remove the vertex corresponding to the outer face. Note that T^* is acyclic (since T was made without adding interior vertices) and connected, so T^* is a tree and has a leaf. This leaf corresponds to a triangle in T where one of its vertices is of degree 2. Remove such a vertex and inductively colour the rest of the graph with 3 colours. Since this vertex has degree 2, we may add it in and use a different colour than its neighbours.

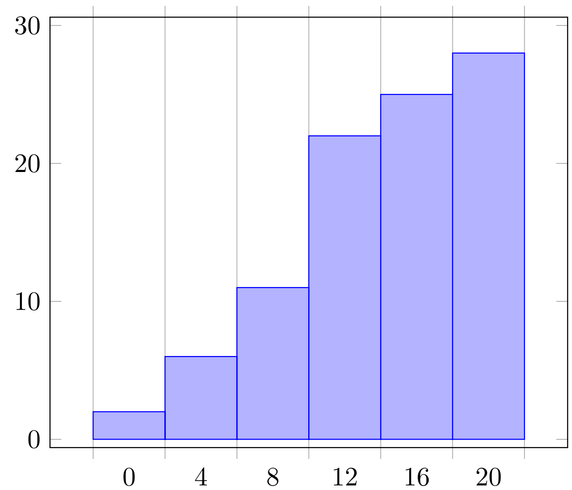
(c). Fix a 3-colouring of T and pick a colour, say red, that is used the least. At most $\lfloor n/3 \rfloor$ vertices are red, and we place cameras on these red vertices. Note that any vertex of a triangle has line-of-sight with the entire triangle. Since a triangle must get different colours on its three vertices, each triangle has a red vertex, hence is guarded by a camera.

(d). Consider a polygon shaped like a comb or a saw. □

Problem	Mean	Stdev
Problem 1 (7 points)	6.50	1.03
Problem 2 (7 points)	3.50	2.83
Problem 3 (7 points)	0.69	2.02
Problem 4 (7 points)	1.81	2.68
Problem 5 (7 points)	0.46	0.95
$\sum$ (35 points total)	12.96	5.07



(a) Histogram of scores.



(b) Cumulative histogram of scores.