# Math 4990 Problem Set 10

*Due Tuesday, Nov 11, 2014 in class*

The Fair Division lecture is based on Chapter 4 of [P], which you are welcome to peruse.

Please try to do the assignment without consulting other sources. If you really must do so, cite your source and write your solution in your own words without copying.

## Assignment

**Problem 1.** Specify some reasonable definition of general position and prove the following:

> Let $X \subset \mathbb{R}^2$ be a set of $n = 6k$ points in general position. There exist three concurrent lines separating $X$ into six groups of $k$ points each.

Remember that the Intermediate Value Theorem applies only to *continuous* functions. Recall that in class, we already saw that six collinear points cannot be partitioned this way. As such, you certainly need to use the general position assumption somewhere!

**Problem 2.** Rays emanating from a common point are **equispaced** if the angles between adjacent rays are all the same. Prove or disprove:

> Every convex polygon $P \subset \mathbb{R}^2$ admits an equipartition into five parts by five equispaced rays.

**Problem 3.** We say $f$ is **adorable** if $f : [0, 1] \to \mathbb{R}$ is a continuous function with $f(0) = f(1) = 0$. The interval $[x, y]$ is a **(horizontal) chord** of length $y - x$ if $f(x) = f(y)$. Let

$$L(f) = \{y - x : f(x) = f(y),\ 0 \le x \le y \le 1\}$$

denote the set of lengths of horizontal chords.

Recall that $\frac{1}{n} \in L(f)$ for any adorable $f$ and $n \in \mathbb{N}$. Prove that this list of lengths is best possible; *i.e.*, prove:

> For every $a \in (0, 1)\backslash\{\frac{1}{n} : n \in \mathbb{N}\}$, there exists an adorable $f$ so that $a \notin L(f)$.

**Problem 4.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose it is **periodic**, *i.e.*, there exists $t > 0$ (called the period) such that $f(x) = f(x+t)$ for all $x$. Show that $f$ has horizontal chords of any length.