Math 5707 Exam 3

Due at the beginning of lecture on Wednesday, May 7, 2014.

Please staple this sheet to the front of your solutions.

There are 6 problems, each worth 7 points. **Turn in solutions for (at most) 5 of them.** If you turn in work for all 6 problems, an arbitrary subset of 5 problems will be graded. Be sure to justify all your work: answers without sufficient justification will receive no credit.

You may use resources such as books and the Internet. Do not collaborate or consult human sources besides the instructor. Clearly indicate any outside sources consulted, and make sure to understand the solutions sufficiently to *explain them in your own words.* Solutions which the instructor views as insignificant alterations of outside sources will receive no credit.

Do not turn in unnecessarily long solutions. Your solutions to each problem should fit within a page (or two) if it were to be typed up. The inclusion of problem statements (if desired) and pictures (which you are encouraged to draw) do not count towards this limit. Do not try to save space by omitting details, truncating sentences, or employing other tricks. Instead, try to find slicker proofs.

Name: ______________________________

ID: ______________________________

Problem 1 (7 points)	
Problem 2 (7 points)	
Problem 3 (7 points)	
Problem 4 (7 points)	
Problem 5 (7 points)	
Problem 6 (7 points)	
$\sum$ (35 points total)	

Problem 1. Let $G = (V, E)$ be a graph on n vertices and $\overline{G}$ its complement (see Diestel §1.1). Prove the following inequalities involving the chromatic numbers $\chi(G)$ and $\chi(\overline{G})$.

(a) $\chi(G) \cdot \chi(\overline{G}) \geq n$.
(b) $\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}$.

[**Hint:** Relate $\chi(\overline{G})$ with $\alpha(G)$, the size of the largest independent set. Use (a) for (b), even if you didn't manage to prove (a).]

Problem 2. For $n \in \mathbb{N}$, prove that there is a tournament on n vertices with at least $n!2^{1-n}$ directed Hamilton paths.

Problem 3. There are n gnomes operating the gnomeship UMN *Enterprise* MATH-5707. This ship has $k \leq n$ different stations, each of which must be occupied by a gnome who is trained for that station. It is desired that **any** subset of k gnomes can operate the ship (in the event that an arbitrary subset of $n-k$ gnomes are ~~killed~~ sleeping). One way to do this is to train all n gnomes for all k stations. However, as training is costly, the Gnome Resources Allocation Provisioning Hegemony asks you to minimize the total number of gnomes trained for the stations. To avoid being executed, you should propose a training scheme and prove that it is optimal, in the sense that it is impossible to have fewer number of total trainings. [**Hint:** The G.R.A.P.H. understands a graph theoretic proof if and only if its relation to the present situation is made explicit.]

Problem 4. For $n \geq 3$, prove that in any edge colouring of a K^n with 2 colours, there is a Hamilton cycle that is the union of two monochromatic paths. [**Hint:** A path is monochromatic if all its edges are coloured equally. A monochromatic Hamilton cycle is the union of two monochromatic paths in many different ways.]

Problem 5. Formulate and prove the generalization of Ford–Fulkerson (Theorem 6.2.2) for networks with multiple source and sink vertices. [**Hint:** In a network, we assumed that a source vertex has only outgoing edges and a sink vertex has only incoming edges. You may wish to redefine or clarify terms such as *flow* (and its *total value*) and *cut* (and its *capacity*). **Do not reinvent the wheel.**]

Problem 6. An orientation is **acyclic** if it contains no (directed) cycles. Let Q_G denote the number of acyclic orientations of a graph G.

(a) Prove that $Q_G = Q_{G-e} + Q_{G/e}$ for each edge $e \in E(G)$.
(b) Prove that $Q_G = |P_G(-1)|$, where $P_G(k)$ is the chromatic polynomial defined in Exercise 5.18 of Diestel.

[**Hint:** In fact, $Q_G = (-1)^{|G|}P_G(-1)$.]