Math 5707 Exam 1

There are 6 problems, each worth 7 points. **Turn in solutions for (at most) 5 of them.** If you turn in work for all 6 problems, an arbitrary subset of 5 problems will be graded. Be sure to justify all your work: answers without sufficient justification will receive no credit.

Grading. The first two major errors cost 3 and 2 points, respectively. Minor errors cost 1 point each. Incoherent solutions are awarded a 0 without regard to the aforementioned scheme. A subset of some common (major, unless otherwise labelled) errors are listed after each problem.

Problem 1. Once upon a time, there was a village with 23 gnomes. Every gnome gave hats to 5 other gnomes. Is it possible that every gnome received hats from the same 5 gnomes to whom he gave hats?

Solution. No. Suppose, towards a contradiction, that "giving hats" is mutual. Then we may construct a graph G whose vertex set is the 23 gnomes and an edge is present between two gnomes if and only if they presented hats to each other as presents. Being 5-regular, the number of edges is evidently $\frac{1}{2} \cdot 23 \cdot 5$, which is absurd. □

Grading. Failing to relate a graph-theoretic setup to the actual problem (minor).

Problem 2. Let $G = (V, E)$ be a graph on $n = |V|$ vertices. Suppose that $G - v$ is a tree for every vertex $v \in V$.

(i) How many edges does G have?
(ii) Determine the structure of G.

Solution. The empty graph is not a tree, so we assume $n \geq 2$. Let $v \in V$ and note that the tree $G - v$ has $n - 1$ vertices hence $n - 2$ edges by the Tree Theorem. There are therefore $n - 2 + d(v)$ edges. Since v is arbitrary, we deduce that G is k-regular. Therefore there are $n - 2 + k = \frac{1}{2}nk$ edges. In other words, $0 = nk - 2n - 2k + 4 = (n - 2)(k - 2)$. If $n = 2$, then G is a single edge or two disjoint vertices. Otherwise, $k = 2$ and G has n edges. Each connected component has a closed Eulerian tour (Theorem 1.8.1), which necessarily is a cycle. So G is a union of disjoint cycles. If there are multiple components, $G - v$ is not a tree (it is neither acyclic nor connected). So $G = C^n$ is a cycle. □

Grading. Claiming that $G - v$ is connected implies G is connected; omitting the case of $n = 2$ or claiming G is 2-connected (minor).

Problem 3. Devise an algorithm to perform the following task. Given a graph $G = (V, E)$, find a subset $S \subseteq V$ of vertices such that the induced subgraph $G[S]$ contains no edges, and that

$$|S| \geq \frac{|V|}{\Delta(G) + 1},$$

where $\Delta(G)$ denotes the maximum degree of the graph G.

Solution. Consider the naïve algorithm:

(i) Let $S = \varnothing$.
(ii) Pick a vertex $v \in V$, add it to S, and delete v and $N(v)$ from V.
(iii) Repeat until $V = \varnothing$.

If $x, y \in S$, and x was added first, then y is not a neighbour of x, lest it be deleted from V prematurely. Furthermore, in each round, we delete at most $\Delta(G) + 1$ vertices from V, so we can run this at least $\frac{|V|}{\Delta(G)+1}$ rounds. □

Grading. Failing to explain why $G[S]$ is edgeless or $|S|$ is big. (As a corollary, stating an algorithm without justification would get at most 2 points.)

Problem 4. Let $n \in \mathbb{N}$ be positive. For each pair of integers x and y such that $1 \leq x \leq y \leq n$, take a card and label one side with x and the other with y.

(i) How many cards are there?

Put these cards on top of each other to form a *deck*, such that sides touching each other have equal labels, *i.e.*, the back of a card has the same label as the front of the next card. You are allowed to flip over the cards when assembling such a deck. Note that a deck of cards have two numbers showing: the front of the first card and the back of the last card. We say that a deck is *orderly* if these two numbers are also equal.

(For example, write $[x, y]$ for a card with x on the front and y on the back. If $n = 3$, the following represents an orderly deck: $[1, 2], [2, 2], [2, 3], [3, 3], [3, 1], [1, 1]$. Note that $[3, 1]$ is the card $[1, 3]$ flipped over.)

(ii) Prove that it is possible to assemble an *orderly* deck using every card if and only if n is odd.

[**Hint:** Relate this to Euler tours of some graph.]

Solution. There are $\binom{n}{1}$ cards with the same number on both sides and $\binom{n}{2}$ cards with different numbers. So there are $\binom{n+1}{2}$ total. We follow the notation in the problem statement.

Consider the complete graph K^n on vertex set $\{1, \ldots, n\}$, which admits a closed Euler tour if and only if n is odd by Theorem 1.8.1. If there is a closed Euler tour $v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$, $v_0 = v_k$, then we can assemble an orderly deck as follows: take $[v_0, v_1], [v_1, v_2], \ldots, [v_{k-1}, v_k]$, and insert $[i, i]$ before the first occurrence of $[i, j]$, $i \neq j$. Conversely, if there is an orderly deck, omit $[i, i]$ for each i to obtain a sequence of cards $[v_0, v_1], [v_1, v_2], \ldots, [v_{k-1}, v_k]$, and construct a closed Euler tour $v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$, where $e_i = \{v_{i-1}, v_i\}$. □

Grading. Miscalculating (i); failing to explain how an Euler tour can be used to construct an orderly deck or *vice versa*; disregarding cards $[x, x]$ in either of the cases above.

Problem 5. (Exercise 2.9 in Diestel.) Let A be a finite set with subsets $A_1, \dots, A_n$, and let $d_1, \dots, d_n \in \mathbb{N}$. Show that there are (pairwise) disjoint subsets $D_k \subseteq A_k$, with $|D_k| = d_k$ for all $k \le n$, if and only if

$$\left| \bigcup_{i \in I} A_i \right| \ge \sum_{i \in I} d_i \tag{*}$$

for all $I \subseteq \{1, \dots, n\}$. [**Hint:** Construct a bipartite graph in which A is one side, and the other side consists of a suitable number of copies of the sets A_i. Define the edge set of the graph so that the desired result can be derived from the marriage theorem.]

Solution. Suppose the D_k exist as above. Then $\bigcup_{i \in I} A_i \supseteq \bigsqcup_{i \in I} D_i$, where the disjoint union of the D_k has cardinality $\sum_{i \in I} d_i$, yielding (*).

Conversely, suppose (*) is satisfied for all $I \subseteq [n] := \{1, \dots, n\}$. We adopt the convention that $[0] := \varnothing$. Let A_k^j be a copy of A_k, for $j \in [d_k]$. Let G be a bipartite graph with A as the set of vertices on the *right*, and $\{A_k^j : k \in [n], j \in [d_k]\}$ as the set of vertices on the *left*. Let $a \in A$ be joined to A_k^j if and only if $a \in A_k^j = A_k$. For a collection S of vertices of the left, let $I = \{i \in [n] : A_i^j \in S \text{ for some } j \in [d_i]\}$, and note that $N(S) = \bigcup_{i \in I} A_i$, so $|N(S)| \ge \sum_{i \in I} d_i$ by (*). Since there are at most d_i copy of each A_i, we have $|S| \le \sum_{i \in I} d_i$, and hence Hall's condition is satisfied. Let M be a complete matching from the left to the right as afforded by Hall's matching theorem. Let $a_k^j \in A$ be the vertex matched with A_k^j, $k \in [n]$, $j \in [d_k]$. Note that the a_k^j are distinct. Then for $k \in [n]$, $D_k = \{a_k^j : j \in [d_k]\}$ are disjoint subsets of A_k with $|D_k| = d_k$, as desired. □

Grading. Incorrect construction of the bipartite graph; insufficient explanation of what a complete matching has to do with the given problem; failure to address one direction of the implication.

Problem 6. (Exercise 2.11 in Diestel.) Let G be a bipartite graph with bipartition $\{A, B\}$. Assume that $\delta(G) \ge 1$, and that $d(a) \ge d(b)$ for every edge ab with $a \in A$. Show that G admits a complete matching from A to B. [**Hint:** Intuitively, the edges between a set $S \subseteq A$ and $N(S)$ create larger degrees in S than in $N(S)$, so they must be spread over more vertices of $N(S)$ than of S. To make this precise, count both S and $N(S)$ as a sum indexed by those edges. Alternatively, consider a minimal set S violating the marriage condition, and count the edges between S and $N(S)$ in two ways.]

Solution. Let $S \subseteq A$, and let $T = N(S)$. Now

$$|S| = \sum_{ab \in E(S,T)} \frac{1}{d(a)} \le \sum_{ab \in E(S,T)} \frac{1}{d(b)} \le \sum_{ab \in E(A,T)} \frac{1}{d(b)} = |T|,$$

where $a \in A$ and $b \in B$ for all three summations. The existence of a complete matching follows from Hall's matching theorem.

Alternatively, suppose, towards a contradiction, that $S \subseteq A$ is minimal such that $|S| > |T|$, with $T = N(S)$. Fix $v \in S$ and note that by minimality of S, there is a complete matching M from $S - v$ to T. As $|T| \le |S| - 1 = |S - v| \le |T|$, actually $|T| = |S| - 1$, and M covers T. Therefore

$$|E(S,T)| = \sum_{a \in S} d(a) = d(v) + \sum_{ab \in M} d(a) > \sum_{ab \in M} d(b) = |E(A,T)| \ge |E(S,T)|,$$

where the strict inequality comes from $\delta(G) \geq 1$, a contradiction. $\square$

Grading. Assuming $E(S,T) = E(A,T)$; not using minimality of S or a matching M in the second proof; not using $\delta(G) \geq 1$.

Problem	Mean	Stdev
Problem 1 (7 points)	6.38	1.97
Problem 2 (7 points)	3.08	2.59
Problem 3 (7 points)	4.63	2.42
Problem 4 (7 points)	4.39	2.73
Problem 5 (7 points)	3.87	2.59
Problem 6 (7 points)	2.61	3.01
$\sum$ (35 points total)	21.13	9.21