Math 5707 Exam 2

There are 6 problems, each worth 7 points. **Turn in solutions for (at most) 5 of them.** If you turn in work for all 6 problems, an arbitrary subset of 5 problems will be graded. Be sure to justify all your work: answers without sufficient justification will receive no credit.

Grading. The first two major errors cost 3 and 2 points, respectively. Minor errors cost 1 point each. Incoherent solutions are awarded a 0 without regard to the aforementioned scheme. A subset of some common (major, unless otherwise labelled) errors are listed after each problem.

Problem 1. Given a graph $G = (V, E)$ with $\delta(G) \geq 2$, prove that there is a connected graph H on the same vertex set V such that $d_G(v) = d_H(v)$ for all $v \in V$.

Solution. Apply induction on the number k of components of G. If $k = 1$, then we are done. Suppose $k > 1$. Let G_1 be a component of G. If G_1 is minimally connected, then it is a tree and has a leaf, contradicting $\delta(G_1) \geq \delta(G) \geq 2$. As such, we may pick an edge $e_1 \in E(G_1)$ such that $G_1 - e_1$ is connected. Similarly, let G_2 be another component of G, and $e_2 \in E(G_2)$ such that $G_2 - e_2$ is connected. Let $e_i = x_i y_i$. Certainly neither $e_3 := x_1 x_2$ nor $e_4 := y_1 y_2$ is an edge of G, lest G_1 and G_2 be connected. Let $H = G - e_1 - e_2 + e_3 + e_4$. Note that $d_G(v) = d_H(v)$ for all $v \in V$. Moreover, $V(G_1) \cup V(G_2)$ now forms a component, while the other components are unaffected. So H has $k - 1$ components. By induction, there is a connected graph H' on V, such that $d_{H'}(v) = d_H(v) = d_G(v)$ for all $v \in V$, as desired. □

Grading. Using an iterated algorithm without addressing termination (minor).

Problem 2. Let G be a planar graph on n vertices. Suppose k is the length of a shortest cycle in G. Prove that G has at most $(n-2)\frac{k}{k-2}$ edges.

Solution. Suppose a planar graph G on n vertices has a shortest cycle of length k. By adding edges if necessary, we may assume that G is connected. Indeed, an upperbound for the number of edges in the new graph implies the same upperbound for the original graph. Fix a drawing of G, and let F be the set of faces. For $f \in F$, the boundary $G[f]$ contains a cycle. Otherwise, $G[f]$ is a forest and it has only one face (Proposition 4.2.4). As such, $G[f] \cup f = \mathbb{R}^2$, so $G = G[f]$ contains no cycles, a contradiction. As all cycles have lengths at least k, we get that $\|G[f]\| \geq k$. Note that f is arbitrary, so this is true for every face.

Let θ be the number of flags (e, f) where e is an edge in the boundary of face f. We have $k\,|F| \leq \theta \leq 2\,|E|$, as each edge $e \in E$ is in the boundary of at most 2 faces, and each face contains at least k edges in its boundary. Substituting $|F| \leq \frac{2}{k}\,|E|$ into Euler's formula $n - |E| + |F| = 2$ gives the desired bound. □

Grading. Applying Euler's formula to a disconnected graph, claiming that every face is bounded by (a graph containing) a cycle, stating $k\,|F| \leq 2\,|E|$ without justification.

Problem 3. For $k, \ell \in \mathbb{N}$ such that $1 \le k \le \ell$, prove that there is a graph G with connectivity $\kappa(G) = k$ and edge-connectivity $\lambda(G) = \ell$.

[**Hint:** See Section 1.4 in Diestel for the definition of edge-connectivity $\lambda(G)$, and note that Proposition 1.4.2 explains why $k \le \ell$ is assumed.]

Solution. Let $k, \ell \in \mathbb{N}$ be given such that $1 \le k \le \ell$. Let G_1 and G_2 be disjoint copies of a $K^{\ell+1}$. Let $A = \{a_1, \dots, a_k\}$ be a subset of k distinct vertices of G_1, and $B = \{b_1, \dots, b_\ell\}$ be a subset of ℓ distinct vertices of G_2. Let $E' = \{a_i b_i : i \in [k]\} \cup \{a_1 b_j : j \in [\ell] \backslash [k]\}$, and consider $G = (G_1 \sqcup G_2) + E'$. Note that $\kappa(G_i) = \delta(G_i) = \ell$, so by Proposition 1.4.2, $\lambda(G_i) = \ell$ as well.

As each edge of E' intersects A, we have that $G - A = (G_1 - A) \sqcup G_2$ is disconnected, so $\kappa(G) \le |A| = k$. To prove equality, delete a set S of at most $k-1$ vertices from G. Each $G_1 - S$ and $G_2 - S$ is connected, as $\kappa(G_i) = \ell > |S|$. Moreover, as $|S| < k$, there exists $i \in [k]$ such that $a_i, b_i \notin S$. Therefore $a_i \in V(G_1 - S)$ and $b_i \in V(G_2 - S)$ are connected by an edge $a_i b_i$ in $G - S$. As such, by transitivity of connectivity, $G - S$ is connected, as desired.

Note that $G - E' = G_1 \sqcup G_2$ is disconnected, so $\lambda(G) \le |E'| = \ell$. To prove equality, delete a set F of at most $\ell - 1$ edges from G. As above, we know that each $G_1 - F$ and $G_2 - F$ is connected, as $\ell(G_i) = \ell > |F|$. Pick an edge $a_i b_j \in E' \backslash F$. Then $a_i \in V(G_1 - F)$ and $b_j \in V(G_2 - F)$ are connected by an edge in $G - F$, so $G - F$ is connected, as desired. □

Grading. Failing to prove one of the four inequalities $\kappa(G) \le k$, $\kappa(G) \ge k$, $\lambda(G) \le \ell$, or $\lambda(G) \ge \ell$.

Problem 4. Let $G = (V, E)$ be a plane graph whose vertices are all on the boundary of the outer face. Prove that there is a partition of V into two sets V_1 and V_2 such that each induced subgraph $G[V_1]$ and $G[V_2]$ is a disjoint union of paths.

[**Hint:** Consider the parity of the distance $d(x, y)$, defined in Section 1.3 of Diestel, from a fixed vertex x.]

Solution. By adding edges if necessary, we may assume that G is connected. For vertices $x, y \in V$, let the distance $d(x, y)$ be the length of a shortest x–y path. Fix $r \in V$. By connectivity, $d(r, x)$ is well-defined, and either odd or even. If $x \in V$ is at an odd distance $d(r, x)$ from r, put $x \in V_1$. Otherwise put $x \in V_2$.

Note that if $G[V_i]$ contains an edge xy, then $d(r, x) = d(r, y)$. Otherwise, suppose $d(r, x) + 2 \le d(r, y)$. Take a shortest r–x path P. Note that $y \notin V(P)$, lest $d(r, y) < d(r, x)$. But then $rPxy$ is an r–y path of length $d(r, x) + 1 < d(r, y)$, a contradiction.

Consider a *connected* subgraph H of some $G[V_i]$, where $|H| > 1$. By the discussion above, each $x \in V(H)$ gives the same $d(r, x)$. As $r \in V_2$ and all its neighbours are in V_1, and hence $r \notin H$. Let P_x be a shortest r–x path. Note that P_x intersects H only at x. If not, and it contains another vertex $y \in V(H)$, then $rP_x y$ is an r–y path shorter than $d(r, x) = d(r, y)$, a contradiction. Let $U = \bigcup_{x \in V(H)} P_x$, and let $H * r$ denote $(H \cup U)/U$, where v_U, the vertex corresponding to the branch set U, is identified with r. Note that $H * r$ is a minor of G.

If $G[V_i]$ contains a $H = K_{1,3}$, then $H * r$ contains a $K_{2,3}$. If $G[V_i]$ contains a $H = C^n$, then $H * r$ contains a K^4 as a minor (contract $n - 2$ contiguous vertices on the cycle C^n). Each case is a contradiction to Exercise 4.22 of Diestel.

Therefore $G[V_i]$ has maximum degree at most 2 and is acyclic, meaning that $G[V_i]$ is a disjoint union of (possibly trivial) paths. □

Grading. Hand-waving instead of, say, citing Exercise 4.22. Given the large number of crucial steps necessary, most other errors, *e.g.*, failing to consider the disconnected case, are considered minor.

Problem 5. For $n \in \mathbb{N}$, prove that there exists a bipartite, 3-regular, planar graph with $2n$ vertices if and only if $n \geq 4$ and $n \neq 5$.

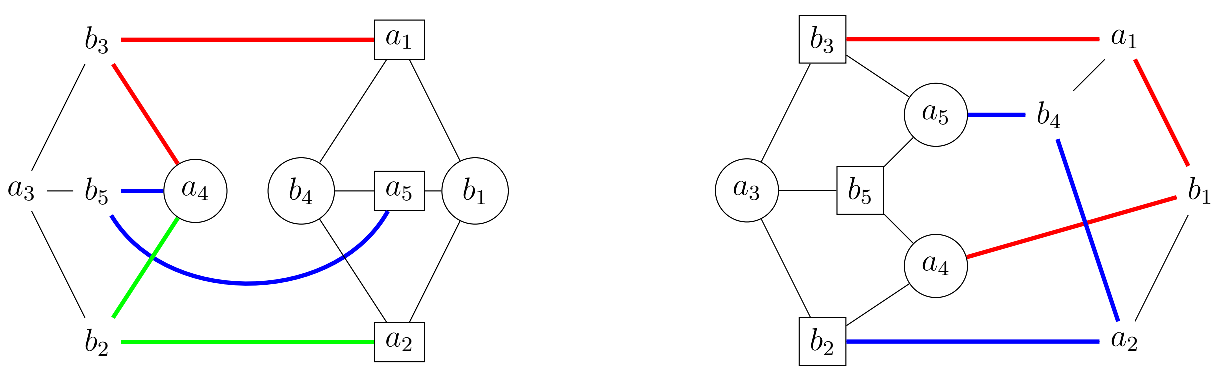
Solution. Suppose $n \geq 4$ is even. Draw $C_1 = x_1x_2 \ldots x_nx_1$ on the plane, and draw $C_2 = y_1y_2 \ldots y_n$ in the interior face of C_1. Add edges x_iy_i in an obvious way to get a 3-regular, planar graph G_n. Moreover, $x_1, x_3, \ldots, x_{n-1}, y_2, y_4, \ldots, y_n$ and its complement forms a bipartition for G_n.

Suppose $n \geq 7$ is odd. Take G_{n-1} defined above, remove edges x_iy_i for $i \in \{1, 3, 5\}$, and draw a new vertex x (resp. y) in the outer (resp. inner) face of C_1 (resp. C_2) and join it to x_1, x_3, x_5 (resp. y_1, y_3, y_5). This gives a bipartite, 3-regular, planar graph.

Conversely, let G be a bipartite, 3-regular graph on $2n$ vertices. As each vertex has degree at most n, we have $n \geq 3$. If $n = 3$, then $G = K_{3,3}$ is not planar by Kuratowski. It remains to show that if $n = 5$, then G contains a $K_{3,3}$ minor, and therefore is not planar.

Note that G contains a C^6. Suppose not, and let $\{M_1, M_2, M_3\}$ be a set of edge-disjoint 1-regular spanning subgraphs of G (1-factorisation, Corollary 2.1.3). As $M_1 \cup M_2$ is 2-regular, it is either $C^4 \sqcup C^6$ or C^{10}. Let $v_1v_2 \ldots v_{10}v_1$ be the 10-cycle. If $v_iv_{i+5} \in M_3$ for any $i \in [5]$ then there is a 6-cycle. Otherwise, WLOG, $v_iv_{i+3} \in M_3$ for all odd i, with index read modulo 10. Then $v_1v_4v_5v_6v_7v_{10}v_1$ is a 6-cycle.

Let $C = a_1b_1a_2b_2a_3b_3a_1$ be a 6-cycle in G. Suppose C is *not* induced, WLOG, C has a chord a_1b_2. Each of $S := \{a_4, a_5, b_4, b_5\}$, being cubic, must be adjacent to (at least) one of $T := \{b_1, a_2, a_3, b_3\}$. As vertices of T are cubic, there are (at most) four S–T edges. The two (in)equalities imply that the remaining eight edges form a cycle on S and a matching between S and T. Contract each of the four S–T edges to get a $K_{3,3}$.



(A) a_5 adjacent to b_1 (B) a_5 adjacent to b_3

FIGURE 1. C is an induced cycle

Otherwise, C is induced, and each of its vertices is adjacent to precisely one vertex in S, and each in S is adjacent to at least one in C, as before. WLOG, a_4 and b_4 are adjacent to two (a_5 and b_5 are adjacent to one) vertices of C, and hence there is a path $a_4b_5a_5b_4$ in G. WLOG, we have edges a_1b_4, a_2b_4, a_3b_5. The remaining three edges depend (only) on whether a_5 is adjacent to b_1. If a_5b_1 is an edge (see Figure 1A), then $\{a_1, a_2, a_5\}$ and $\{b_1, b_4, a_4\}$ form branch vertices of a topological $K_{3,3}$ minor, with subdivided paths $a_1b_3a_4$, $a_2b_2a_4$, and $a_5b_5a_4$. Otherwise, WLOG, a_5b_3 is an edge (see Figure 1B), then $\{a_3, a_4, a_5\}$ and $\{b_2, b_3, b_5\}$ form branch vertices of a topological $K_{3,3}$ minor, with subdivided paths $a_4b_1a_1b_3$ and $a_5b_4a_2b_2$. □

Grading. Hand-waving the $n = 5$ case, especially if not using Kuratowski; assuming all boundaries are cycles.

Problem 6. For $k \in \mathbb{N}$, let $G = (V, E)$ be a k-connected graph. Suppose $f : V \to \mathbb{Z}$ is a function with integer values such that $\sum_{v\in V} f(v) = 0$ and $\sum_{v\in V} |f(v)| = 2k$, where $|x|$ is the absolute value of x. Prove that there are k independent paths such that $|f(v)|$ of them have v as an end for each $v \in V$.

Solution. Apply induction on k, which has trivial base case $k = 0$. For the inductive step, let G' be an IG obtained from G by replacing each vertex $x \in V(G)$ with a branch set V_x, where $G'[V_x]$ is a complete graph of order $\max\{1, |f(x)|\}$, and *all* V_x–V_y edges are present if $xy \in E(G)$.

Note that G' is k-connected. Indeed, take $S' \subseteq V(G')$ with $|S'| < k$. Contract $G' - S'$ along the branch sets $V_x \backslash S'$. The result is an induced subgraph $G - S$ of G, where $x \in S$ if and only if $V_x \subseteq S'$. As such, $|S| \leq |S'| < k$ and, as G is k-connected, $G - S$ is connected. As contraction preserves connectedness (contracting an edge does not alter the number of connected components), $G' - S'$ is connected, as desired.

Let $A = \{v \in V : f(v) > 0\}$ and $B = \{v \in V : f(v) < 0\}$. Define $A' = \bigsqcup_{a\in A} V_a$ and $B' = \bigsqcup_{b\in B} V_b$. Note that

$$|A'| + |B'| = \sum_{a\in A} |f(a)| + \sum_{b\in B} |f(b)| = 2k$$

and

$$|A'| - |B'| = \sum_{a\in A} f(a) + \sum_{b\in B} f(b) = 0,$$

so $|A'| = |B'| = k$. By Menger's theorem, as G' is k-connected, there is a set of k disjoint A'–B' paths in G'. Note that each vertex of A' and B' is used. Contracting G' along the V_x then naturally maps these k disjoint A'–B' paths to k *independent* A–B paths, where each vertex $x \in A \sqcup B$ has $|V_x| = |f(x)|$ paths ending at it. As their interior vertices are disjoint singleton branch sets, the paths with interior vertices remain distinct. For paths that are single edges, it is possible that multiple edges between branch sets V_x and V_y are chosen. If this is not the case, then we are done.

It remains to consider the case when there is an edge $xy \in E(G)$ such that $f(x) > 1$, $f(y) < -1$. By Menger's theorem, G has k independent paths between any two vertices, so $G - xy$ has $k-1$ independent paths between any two vertices, and therefore $G-xy$ is $(k-1)$-connected by Menger's theorem again. Modify f by decreasing $f(x)$ by 1 and increasing $f(y)$ by 1 to get a new function f', which satisfies $\sum_{v\in V} f'(v) = 0$ and $\sum_{v\in V} |f'(v)| = 2(k - 1)$. By induction, $G - xy$ has $k - 1$ independent paths such that $|f'(v)|$ of them have v as an end for each $v \in V$. As the path xy has no interior vertices, it may be added to the set to form k independent paths of G, as desired. □

Grading. Failure to consider edge ab with $f(a) > 1$ and $f(b) < -1$.

Problem	Mean	Stdev	Mode
Problem 1 (7 points)	4.71	3.00	7
Problem 2 (7 points)	1.82	1.51	2
Problem 3 (7 points)	5.07	3.05	7
Problem 4 (7 points)	2.57	2.44	0
Problem 5 (7 points)	3.50	2.73	4
Problem 6 (7 points)	1.50	2.55	0
$\sum$ (35 points total)	16.06	9.12	

Counting multiplicities of $\{0, 1, \ldots, 7\}$ for all problems and students, the most common score for a problem is 0, followed by 7, 4, and 2, in that order, indicating 0, 1, and 2 major errors committed, respectively. Other scores are less frequently assigned.