EXAMPLE 8.50 **Illustrating Case 1**

Let $A_1 = \{a, b\}$, $A_2 = \{b, c\}$, $A_3 = \{c, d\}$, and $A_4 = \{a, d\}$. Then $|A_i| \geq 2$ for $i = 1, 2, 3, 4$. Also, $|A_i \cup A_j| \geq 3$ whenever $1 \leq i < j \leq 4$. In addition, $|A_i \cup A_j \cup A_k| \geq 4$ for all choices of i, j, k with $1 \leq i < j < k \leq 4$. Finally, notice that $|A_1 \cup A_2 \cup A_3 \cup A_4| = 4$. Consequently, both the marriage condition and the enhanced marriage condition are satisfied.

Let $r_4 = a$. Then $B_1 = \{b\}$, $B_2 = \{b, c\}$, and $B_3 = \{c, d\}$. It is easy to verify that $\{B_1, B_2, B_3\}$ satisfies the marriage condition, so the inductive hypothesis guarantees a system of distinct representatives. In this example there is only one such system: $r_1 = b$, $r_2 = c$, and $r_3 = d$.

The list r_1, r_2, r_3, r_4 does form a system of distinct representatives for $\{A_1, A_2, A_3, A_4\}$. ■

EXAMPLE 8.51 **Illustrating Case 2**

Let $A_1 = \{a, b\}$, $A_2 = \{b, c\}$, $A_3 = \{a, b\}$, and $A_4 = \{c, d, e\}$.

Since $|A_1 \cup A_3| = 2$, the enhanced marriage condition is not satisfied. However, it is easy to verify that the marriage condition *is* satisfied.

Rename the sets so that $A_1 = \{a, b\}$, $A_2 = \{a, b\}$, $A_3 = \{b, c\}$, and $A_4 = \{c, d, e\}$ (changing subscripts $i_1 = 1$ and $i_2 = 3$ to subscripts 1 and 2). The inductive hypothesis asserts the existence of a system of distinct representatives for $\{A_1, A_2\}$. One such system is $r_1 = a$ and $r_2 = b$. Notice that $m = n - 2$, so complete induction is necessary.

Removing $S = \{a, b\}$ from A_3 and A_4 results in $B_3 = \{c\}$ and $B_4 = \{c, d, e\}$. The collection $\{B_3, B_4\}$ satisfies the marriage condition and has less than 4 members, so a system of distinct representatives exists (by the inductive hypothesis). One such system is $r_3 = c$ and $r_4 = d$.

The list r_1, r_2, r_3, r_4 forms a system of distinct representatives for $\{A_1, A_2, A_3, A_4\}$. ■