6.1 Definitions in Chapter 6

6.1 Sample Space, Outcome
6.2 Event
6.3 Complement of an Event
6.4 Mutually Exclusive Events
6.5 The Probabilities of Outcomes and Events
6.6 Equally Likely Outcomes
6.7 Vowels; Consonants
6.8 Conditional Probability
6.9 Independent Events
6.10 Value of an Outcome
6.11 Random Variable
6.12 Expected Value
6.13 Odds
6.14 Fair Game
6.15 Binomial Distribution
6.20 Geometric Distribution

6.2 Probability Principles

<table>
<thead>
<tr>
<th>Probability Principles</th>
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<tbody>
<tr>
<td>In the formulas that follow, assume that $A$ and $B$ are events.</td>
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<tr>
<td>1. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$</td>
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<tr>
<td>2. $P(A \cap B) = 0$, if $A$ and $B$ are mutually exclusive events</td>
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<tr>
<td>3. $P(A \cup B) = P(A) + P(B)$, if $A$ and $B$ are mutually exclusive events</td>
</tr>
<tr>
<td>4. $P(A \cap B) = P(A) \cdot P(B</td>
</tr>
<tr>
<td>5. $P(A \cap B) = P(A) \cdot P(B)$, if $A$ and $B$ are independent events</td>
</tr>
<tr>
<td>6. $P(\overline{A}) = 1 - P(A)$</td>
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<td>7. $P(B</td>
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</table>

6.3 Sample Exam Questions

1. Translate the following concepts into their set-theoretic counterparts.
   (a) Sample space
   (b) Outcome
   (c) Event
   (d) Complement of an event

2. Define the following terms.
   (a) Mutually exclusive events
   (b) Independent events
   (c) Random variable
3. Suppose that \( P(A) = 0.4, P(B) = 0.5, \) and \( A \cap B = \emptyset. \)
   (a) What can be determined about \( P(A \cap B)? \)
   (b) What can be determined about \( P(A \cup B)? \)

4. Suppose that \( P(A) = 0.3, P(\overline{B}) = 0.6, \) and \( A \cap B \neq \emptyset. \)
   (a) What can be determined about \( B? \)
   (b) What can be determined about \( P(A \cup B)? \)

5. What is the probability of being dealt a 4-card hand from a standard 52-card deck having two cards of the same kind (i.e.,
   two sevens, or two queens, etc.) and the remaining two cards
   being different kinds (from each other and from the matching
   pair)?

6. A random experiment consists of choosing a random card from a standard 52-card deck. Let \( H \) be the event “a heart”
   and \( N \) be the event “a non-face card” (i.e., a card in \([\text{ace}, 2, 3, \ldots , 10]\)).
   (a) Calculate \( P(H | N). \)
   (b) Calculate \( P(N | H). \)
   (c) Are the events \( H \) and \( N \) independent?
   (d) Are the events \( H \) and \( N \) mutually exclusive?

7. Suppose that \( P(A) = 0.3, P(B) = 0.6, \) and \( P(A \mid B) = 0.4. \)
   (a) Calculate \( P(A \cap B). \)
   (b) Calculate \( P(B \mid A). \)

8. A random experiment consists of rolling an unfair, six-sided die. The digit 6 is three times as likely to appear as the numbers
   2 and 4. The numbers 2 and 4 are twice as likely to appear as one of the numbers, 1, 3, and 5.
   (a) Assign appropriate probabilities to the six outcomes in
   the sample space.

9. A lottery has three levels of prizes. First prize is \$100, sec-
   ond prize is \$50, and third prize is \$25. The \( S : T \) odds of
   winning are \( 2 : 1000, 5 : 800, \) and \( 12 : 400, \) respectively.
   (a) What are the probabilities of winning the various levels
   of prizes?
   (b) What is the probability of not winning a prize?
   (c) What is a fair price for this lottery?

10. State and prove Bayes’s theorem.

11. In a certain math class, there is a probability of \( \frac{35}{47} \)
   that a randomly chosen student is male. The probability that a
   randomly chosen female is a computer science major is \( \frac{1}{7}. \)
   The probability that a randomly chosen male is a computer
   science major is \( \frac{23}{35}. \)
   What is the probability that a randomly
   chosen computer science major is female?

12. The probability that a widget, coming off the factory line, is
   defective is 0.08. Assuming the occurrences of defects are in-
   dependent, what is the probability that there are more than
   2 defective items in a sample of size 20?
   (b) Suppose that the random variable, \( X, \) is assigned the
   value of the digit that appears when the die is rolled.
   What is the expected value of \( X? \)

6.4 Projects

Mathematics

1. Write a brief expository report about the basic probability
   definitions and concepts when the sample space is a subset of
   the real numbers. How are events defined? How is the
   probability of an event calculated? What is different about
   the probabilities of outcomes in this setting as compared to
   finite probability theory? What is the probability model?

2. Write a brief expository report about Bayesian statistics.
   How is Bayes’s theorem connected to Bayesian statistics?

Computer Science

1. Write a program that accepts a set of odds (in \( S : T \) for-
   mat) and the corresponding set of prize values. The program
   should calculate the event probabilities and the ex-
   pected value. The program should check that the probabil-
   ities sum to a number that is very close to 1. Use a graphical
   user interface for both input and output.

2. Write a program with a graphical user interface that accepts
   the information needed to use Theorem 6.26. The program
   should calculate \( P(B_i \mid A), \) for each \( i. \) It should also check
   that all the required information is available and that the sum
   of the \( B_i \) probabilities is very close to 1.

General

1. Write a brief expository paper about the use of Bayes’s theo-
   rem in the philosophy of science.

2. Write a brief expository paper about the use of Bayes’s theo-
   rem in philosophy (excluding the philosophy of science).

6.5 Solutions to Sample Exam Questions

1. (a) A sample space is a set that is in the role of a universal
   set.
   (b) An outcome is an element of the universal set.
   (c) An event is a subset of the universal set.
   (d) The complement of an event is its set complement.

2. (a) See Definition 6.4 on page 276.
   (b) See Definition 6.9 on page 285.
   (c) See Definition 6.11 on page 303.

3. (a) Since \( A \cap B = \emptyset, P(A \cap B) = 0. \)
   (b) Since \( A \cap B = \emptyset, \) the events, \( A \) and \( B, \) are mutually
   exclusive. Therefore, probability principle 3 applies.

4. (a) \( P(A \cup B) = P(A) + P(B) = .9. \)
   (b) According to probability principle 1,
   \[ P(A \cup B) = P(A) + P(B) - P(A \cap B) = .3 + .4 - P(A \cap B). \]

   More information would be needed to say more.

5. There are \( C(13,1) \) ways to choose the kind of card to re-
   peat, and \( C(4,2) \) ways to choose the two cards of the same
   kind. There are \( C(12,2) \) ways to choose which two kinds of
cards will finish the hand. For each of these kinds, there are $C(4, 1)$ ways to pick one of the four cards of that kind. The probability of the hand is therefore
\[
\frac{C(13, 1) \cdot C(4, 2) \cdot C(12, 2) \cdot C(4, 1) \cdot C(4, 1)}{C(52, 4)}
\]
\[
= \frac{13 \cdot \frac{4!}{2!2!} \cdot \frac{12!}{10!} \cdot 4 \cdot 4}{48!}
\]
\[
= \frac{13 \cdot 6 \cdot 66 \cdot 4 \cdot 4}{13 \cdot 17 \cdot 25 \cdot 49}
\]
\[
= \frac{6,336}{20,825} \simeq .304.
\]
6. (a) There are 12 face cards, so there are 52 – 12 = 40 nonface cards. One-fourth of them are hearts, so $P(H | N) = .25$.
(b) There are 13 hearts, with 10 nonface cards that are hearts. Thus, $P(N | H) = \frac{10}{13}$.
(c) The events are independent since $P(H | N) = .25 = P(H)$. [Alternatively, $P(N | H) = \frac{10}{13} \neq \frac{40}{52} = P(N)$.] (d) They are not mutually exclusive, since, for example, the outcome “4 of hearts” is in both events.
7. (a) Using probability principle 4,
\[
P(A \cap B) = P(B) \cdot P(A | B) = (.6)(.4) = .24.
\]
(b) Using probability principle 4 and part (a),
\[
P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{.24}{.3} = .8.
\]
8. (a) If the odd numbers are assigned probability $x$, then 2 and 4 have probability 2$x$ and 6 has probability 3(2$x$) = 6$x$. Since the sum of the probabilities must be 1, $x + x + x + 2x + 2x + 6x = 1$. This implies that $x = \frac{1}{13}$. The following table summarizes the probabilities.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(x)$</td>
<td>$\frac{1}{13}$</td>
<td>$\frac{2}{13}$</td>
<td>$\frac{3}{13}$</td>
<td>$\frac{1}{13}$</td>
<td>$\frac{2}{13}$</td>
<td>$\frac{6}{13}$</td>
</tr>
</tbody>
</table>

(b) The expected value is
\[
1 \cdot \frac{1}{13} + 2 \cdot \frac{2}{13} + 3 \cdot \frac{3}{13} + 4 \cdot \frac{2}{13} + 5 \cdot \frac{1}{13} + 6 \cdot \frac{6}{13} = \frac{57}{13} \simeq 4.38.
\]
9. (a) $P($first prize$) = \frac{2}{5000} = \frac{1}{2500}$
$P($second prize$) = \frac{5}{500} = \frac{1}{100}$
$P($third prize$) = \frac{12}{500} = \frac{3}{125}$
(b) $P($losing$) = 1 - \frac{1}{500} - \frac{1}{160} - \frac{3}{100} = \frac{3,847}{4,000} \simeq .962$.
(c) The expected value gives a fair price. That value is
\[
100 \cdot \frac{1}{500} + 50 \cdot \frac{1}{160} + 25 \cdot \frac{3}{100} + 0 \cdot \frac{3,847}{4,000} = \frac{101}{80} = 1.2625.
\]
A fair price would be $1.26 per ticket (assuming the anticipated number of people enter the lottery). Note that this ignores the expense of running the lottery.
10. Bayes’s Theorem: Let $A$ and $B$ be events. Then
\[
P(B \mid A) = \frac{P(B) \cdot P(A \mid B)}{P(A)} = \frac{P(B) \cdot P(A \mid B)}{P(B) \cdot P(A \mid B) + P(\overline{B}) \cdot P(A \mid \overline{B})}.
\]
**Proof:**
probability principle 4 implies that
\[
P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B) \cdot P(A \mid B)}{P(B) \cdot P(A \mid B) + P(\overline{B}) \cdot P(A \mid \overline{B})}.
\]
Since $B$ and $\overline{B}$ are disjoint, $A = (A \cap B) \cup (A \cap \overline{B})$. Probability principle 3 implies that
\[
P(A) = P(A \cap B) + P(A \cap \overline{B}).
\]
Hence,
\[
P(B \mid A) = \frac{P(B) \cdot P(A \mid B)}{P(B) \cdot P(A \mid B) + P(\overline{B}) \cdot P(A \mid \overline{B})}.
\]

11. This is easy to solve using Theorem 6.25. Let $M$ represent the event “male” and $F$ represent the event “female.” Also, let $C$ represent the event “computer science major.” The information given can be written as $P(M) = \frac{35}{47}$, $P(C \mid F) = \frac{1}{4}$ and $P(C \mid M) = \frac{3}{47}$. Notice that $P(F) = 1 - P(M) = \frac{12}{47}$. Bayes’s theorem implies that
\[
P(F \mid C) = \frac{P(F) \cdot P(C \mid F)}{P(F) \cdot P(C \mid F) + P(M) \cdot P(C \mid M)}
\]
\[
= \frac{12}{47} \frac{1}{4} \frac{35}{47} \frac{23}{35}
\]
\[
= \frac{3}{26} \simeq .115.
\]
12. The binomial distribution with $p = .08$ and $n = 20$ will be handy. Let $X$ count the number of defects in the sample.
\[
P(X \geq 2) = 1 - P(X \leq 1)
= P(X = 0) + P(X = 1)
= 1 - \left( \frac{20}{0} \right) .08^0 \cdot .92^{20} + \left( \frac{20}{1} \right) .08^1 \cdot .92^{19}
\]
\[
\simeq .0483
\]