CHAPTER

6 Review

6.1 Definitions in Chapter 6

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6.2 Probability Principles

Probability Principles

In the formulas that follow, assume that A and B are events.

1. $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$
2. $\mathbf{P}(A \cap B) = 0$, if A and B are mutually exclusive events
3. $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B)$, if A and B are mutually exclusive events
4. $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B \mid A) = \mathbf{P}(B) \cdot \mathbf{P}(A \mid B)$
5. $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$, if A and B are independent events
6. $\mathbf{P}(\overline{A}) = 1 - \mathbf{P}(A)$
7. $\mathbf{P}(B \mid A) = \frac{\mathbf{P}(B)}{\mathbf{P}(A)}$, if $A \cap B = B$

6.3 Sample Exam Questions

1. Translate the following concepts into their set-theoretic counterparts.

(a) Sample space (b) Outcome (c) Event

(d) Complement of an event

2. Define the following terms.

(a) Mutually exclusive events

(b) Independent events

(c) Random variable

3. Suppose that $\mathbf{P}(A) = 0.4$, $\mathbf{P}(B) = 0.5$, and $A \cap B = \emptyset$.
   (a) What can be determined about $\mathbf{P}(A \cap B)$?
   (b) What can be determined about $\mathbf{P}(A \cup B)$?
4. Suppose that $\mathbf{P}(A) = 0.3$, $\mathbf{P}(\overline{B}) = 0.6$, and $A \cap B \neq \emptyset$.
   (a) What can be determined about B?
   (b) What can be determined about $\mathbf{P}(A \cup B)$?
5. What is the probability of being dealt a 4-card hand from a standard 52-card deck having two cards of the same kind (i.e., two sevens, or two queens, etc.) and the remaining two cards being different kinds (from each other and from the matching pair)?
6. A random experiment consists of choosing a random card from a standard 52-card deck. Let H be the event "a heart" and N be the event "a non-face card" (i.e., a card in {ace, 2, 3, . . . , 10}).
   (a) Calculate $\mathbf{P}(H \mid N)$. (b) Calculate $\mathbf{P}(N \mid H)$.
   (c) Are the events H and N independent?
   (d) Are the events H and N mutually exclusive?
7. Suppose that $\mathbf{P}(A) = 0.3$, $\mathbf{P}(B) = 0.6$, and $\mathbf{P}(A \mid B) = 0.4$.
   (a) Calculate $\mathbf{P}(A \cap B)$. (b) Calculate $\mathbf{P}(B \mid A)$.
8. A random experiment consists of rolling an unfair, six-sided die. The digit 6 is three times as likely to appear as the numbers 2 and 4. The numbers 2 and 4 are twice as likely to appear as one of the numbers, 1, 3, and 5.
   (a) Assign appropriate probabilities to the six outcomes in the sample space.
   (b) Suppose that the random variable, X, is assigned the value of the digit that appears when the die is rolled. What is the expected value of X?
9. A lottery has three levels of prizes. First prize is \$100, second prize is \$50, and third prize is \$25. The $S : T$ odds of winning are $2 : 1000$, $5 : 800$, and $12 : 400$, respectively.
   (a) What are the probabilities of winning the various levels of prizes?
   (b) What is the probability of not winning a prize?
   (c) What is a fair price for this lottery?
10. State and prove Bayes's theorem.
11. In a certain math class, there is a probability of $\frac{35}{47}$ that a randomly chosen student is male. The probability that a randomly chosen female is a computer science major is $\frac{1}{4}$. The probability that a randomly chosen male is a computer science major is $\frac{23}{35}$. What is the probability that a randomly chosen computer science major is female?
12. The probability that a widget, coming off the factory line, is defective is 0.08. Assuming the occurrences of defects are independent, what is the probability that there are more at least 2 defective items in a sample of size 20?

6.4 Projects

Mathematics

1. Write a brief expository report about the basic probability definitions and concepts when the sample space is a subset of the real numbers. How are events defined? How is the probability of an event calculated? What is different about the probabilities of outcomes in this setting as compared to finite probability theory? What is the probability model?
2. Write a brief expository report about Bayesian statistics. How is Bayes's theorem connected to Bayesian statistics?

Computer Science

1. Write a program that accepts a set of odds (in $S : T$ format) and the corresponding set of prize values. The program should calculate the event probabilities and the expected value. The program should check that the probabilities sum to a number that is very close to 1. Use a graphical user interface for both input and output.
2. Write a program with a graphical user interface that accepts the information needed to use Theorem 6.26. The program should calculate $\mathbf{P}(B_i \mid A)$, for each i. It should also check that all the required information is available and that the sum of the B_i probabilities is very close to 1.

General

1. Write a brief expository paper about the use of Bayes's theorem in the philosophy of science.
2. Write a brief expository paper about the use of Bayes's theorem in philosophy (excluding the philosophy of science).

6.5 Solutions to Sample Exam Questions

1. (a) A sample space is a set that is in the role of a universal set.
   (b) An outcome is an element of the universal set.
   (c) An event is a subset of the universal set.
   (d) The complement of an event is its set complement.
2. (a) See Definition 6.4 on page 276.
   (b) See Definition 6.9 on page 285.
   (c) See Definition 6.11 on page 303.
3. (a) Since $A \cap B = \emptyset$, $\mathbf{P}(A \cap B) = 0$.
   (b) Since $A \cap B = \emptyset$, the events, A and B, are mutually exclusive. Therefore, probability principle 3 applies.
   $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) = .9$.
4. (a) $\mathbf{P}(B) = 1 - \mathbf{P}(\overline{B}) = .4$, according to probability principle 6.
   (b) According to probability principle 1,
   $$\begin{aligned}\mathbf{P}(A \cup B) &= \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)\\ &= .3 + .4 - \mathbf{P}(A \cap B).\end{aligned}$$
   More information would be needed to say more.
5. There are $C(13, 1)$ ways to choose the kind of card to repeat, and $C(4, 2)$ ways to choose the two cards of the same kind. There are $C(12, 2)$ ways to choose which two kinds of

cards will finish the hand. For each of these kinds, there are $C(4, 1)$ ways to pick one of the four cards of that kind. The probability of the hand is therefore

$$\frac{C(13,1)\cdot C(4,2)\cdot C(12,2)\cdot C(4,1)\cdot C(4,1)}{C(52,4)}$$

$$= \frac{13\cdot\frac{4!}{2!\cdot 2!}\cdot\frac{12!}{2!\cdot 10!}\cdot 4\cdot 4}{\frac{52!}{4!\cdot 48!}}$$

$$= \frac{13\cdot 6\cdot 66\cdot 4\cdot 4}{13\cdot 17\cdot 25\cdot 49}$$

$$= \frac{6,336}{20,825} \simeq .304.$$

6. (a) There are 12 face cards, so there are $52 - 12 = 40$ nonface cards. One-fourth of them are hearts, so $\mathbf{P}(H \mid N) = .25$.

(b) There are 13 hearts, with 10 nonface cards that are hearts. Thus, $\mathbf{P}(N \mid H) = \frac{10}{13}$.

(c) The events are independent since $\mathbf{P}(H \mid N) = .25 = \mathbf{P}(H)$. [Alternatively, $\mathbf{P}(N \mid H) = \frac{10}{13} = \frac{40}{52} = \mathbf{P}(N)$.]

(d) They are not mutually exclusive, since, for example, the outcome "4 of hearts" is in both events.

7. (a) Using probability principle 4,

$$\mathbf{P}(A\cap B) = \mathbf{P}(B)\cdot\mathbf{P}(A \mid B) = (.6)(.4) = .24.$$

(b) Using probability principle 4 and part (a),

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(A\cap B)}{\mathbf{P}(A)} = \frac{.24}{.3} = .8.$$

8. (a) If the odd numbers are assigned probability x, then 2 and 4 have probability $2x$ and 6 has probability $3(2x) = 6x$. Since the sum of the probabilities must be 1, $x + x + x + 2x + 2x + 6x = 1$. This implies that $x = \frac{1}{13}$. The following table summarizes the probabilities.

x	1	2	3	4	5	6
$\mathbf{P}(x)$	$\frac{1}{13}$	$\frac{2}{13}$	$\frac{1}{13}$	$\frac{2}{13}$	$\frac{1}{13}$	$\frac{6}{13}$

(b) The expected value is

$$1\cdot\frac{1}{13} + 2\cdot\frac{2}{13} + 3\cdot\frac{1}{13} + 4\cdot\frac{2}{13} + 5\cdot\frac{1}{13} + 6\cdot\frac{6}{13} = \frac{57}{13} \simeq 4.38.$$

9. (a) $\mathbf{P}(\text{first prize}) = \frac{2}{1,000} = \frac{1}{500}$
$\mathbf{P}(\text{second prize}) = \frac{5}{800} = \frac{1}{160}$
$\mathbf{P}(\text{third prize}) = \frac{12}{400} = \frac{3}{100}$

(b) $\mathbf{P}(\text{losing}) = 1 - \frac{1}{500} - \frac{1}{160} - \frac{3}{100} = \frac{3,847}{4,000} \simeq .962.$

(c) The expected value gives a fair price. That value is

$$100\cdot\frac{1}{500} + 50\cdot\frac{1}{160} + 25\cdot\frac{3}{100} + 0\cdot\frac{3,847}{4,000}$$

$$= \frac{101}{80} = 1.2625.$$

A fair price would be \$1.26 per ticket (assuming the anticipated number of people enter the lottery). Note that this ignores the expense of running the lottery.

10. Bayes's Theorem: Let A and B be events. Then

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(B)\cdot\mathbf{P}(A \mid B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(B)\cdot\mathbf{P}(A \mid B)}{\mathbf{P}(B)\cdot\mathbf{P}(A \mid B) + \mathbf{P}(\overline{B})\cdot\mathbf{P}(A \mid \overline{B})}.$$

Proof:
probability principle 4 implies that

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(A\cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(B)\cdot\mathbf{P}(A \mid B)}{\mathbf{P}(A)}.$$

Since B and $\overline{B}$ are disjoint, $A = (A\cap B)\cup(A\cap\overline{B})$. Probability principle 3 implies that

$$\mathbf{P}(A) = \mathbf{P}(A\cap B) + \mathbf{P}(A\cap\overline{B}).$$

Hence,

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(B)\cdot\mathbf{P}(A \mid B)}{\mathbf{P}(A\cap B) + \mathbf{P}(A\cap\overline{B})} = \frac{\mathbf{P}(B)\cdot\mathbf{P}(A \mid B)}{\mathbf{P}(B)\cdot\mathbf{P}(A \mid B) + \mathbf{P}(\overline{B})\cdot\mathbf{P}(A \mid \overline{B})}. \qquad \square$$

11. This is easy to solve using Theorem 6.25. Let M represent the event "male" and F represent the event "female." Also, let C represent the event "computer science major." The information given can be written as $\mathbf{P}(M) = \frac{35}{47}$, $\mathbf{P}(C \mid F) = \frac{1}{4}$ and $\mathbf{P}(C \mid M) = \frac{23}{35}$. Notice that $\mathbf{P}(F) = 1 - \mathbf{P}(M) = \frac{12}{47}$. Bayes's theorem implies that

$$\mathbf{P}(F \mid C) = \frac{\mathbf{P}(F)\cdot\mathbf{P}(C \mid F)}{\mathbf{P}(F)\cdot\mathbf{P}(C \mid F) + \mathbf{P}(M)\cdot\mathbf{P}(C \mid M)}$$

$$= \frac{\frac{12}{47}\cdot\frac{1}{4}}{\frac{12}{47}\cdot\frac{1}{4} + \frac{35}{47}\cdot\frac{23}{35}}$$

$$= \frac{\frac{3}{47}}{\frac{3}{47} + \frac{23}{47}}$$

$$= \frac{3}{26} \simeq .115.$$

12. The binomial distribution with $p = 0.08$ and $n = 20$ will be handy. Let X count the number of defects in the sample.

$$\begin{aligned}\mathbf{P}(X \geq 2) &= 1 - \mathbf{P}(X \leq 1)\\ &= \mathbf{P}(X = 0) + \mathbf{P}(X = 1)\\ &= 1 - \left(\binom{20}{0}0.08^0\cdot 0.92^{20} + \binom{20}{1}0.08^1\cdot 0.92^{19}\right)\\ &\simeq 0.483\end{aligned}$$