CHAPTER 7

Review

7.1 Definitions in Chapter 7

7.1 Recursion

7.3 Recursively Defined Functions

7.4 Recurrence Relation

7.5 Homogeneous; Constant Coefficients; Linear

7.6 Linear Homogeneous Recurrence Relations with Constant Coefficients of Degree *k*

7.7 The Characteristic Equation

7.15 Nondecreasing Function

7.20 Generating Function

7.23 The Derivative of a Generating Function

7.24 Generalized Binomial Coefficients

7.2 Some Useful Generating Functions

$G(z)$	Summation Notation	Expanded Notation
$\dfrac{1}{1-z}$	$\sum_{k=0}^{\infty} z^k$	$1 + z + z^2 + z^3 + \cdots$
$\dfrac{1}{1+z}$	$\sum_{k=0}^{\infty} (-1)^k z^k$	$1 - z + z^2 - z^3 + \cdots$
$\dfrac{1}{1-z^m}$	$\sum_{k=0}^{\infty} z^{mk}$	$1 + z^m + z^{2m} + z^{3m} + \cdots$
$\dfrac{1}{1-cz}$	$\sum_{k=0}^{\infty} c^k z^k$	$1 + cz + c^2z^2 + c^3z^3 + \cdots$
$\dfrac{1}{(1-z)^m}$	$\sum_{k=0}^{\infty} \binom{m+k-1}{k} z^k$	$1 + mz + \binom{m+1}{2} z^2 + \binom{m+2}{3} z^3 + \cdots$
$\dfrac{z}{(1-z)^2}$	$\sum_{k=0}^{\infty} k z^k$	$0 + z + 2z^2 + 3z^3 + \cdots$
$(1+z)^c$	$\sum_{k=0}^{\infty} \binom{c}{k} z^k$	$1 + cz + \binom{c}{2} z^2 + \binom{c}{3} z^3 + \cdots$
e^z	$\sum_{k=0}^{\infty} \dfrac{1}{k!} z^k$	$1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$

7.3 Sample Exam Questions

1. Create a recursive algorithm that calculates $\binom{n}{r}$ for integers, n and r, with $0 \leq r \leq n$. Use Pascal's theorem: $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

2. Recall the recursive definition of the Sierpinski curves S_n. The recursions follow.

$$\mathrm{S}_n : \mathrm{A}_n \searrow \mathrm{B}_n \swarrow \mathrm{C}_n \nwarrow \mathrm{D}_n \nearrow$$

$$\mathrm{A}_n : \mathrm{A}_{n\text{-}1} \searrow \mathrm{B}_{n\text{-}1} \rightarrow\rightarrow \mathrm{D}_{n\text{-}1} \nearrow \mathrm{A}_{n\text{-}1}$$

$$\mathrm{B}_n : \mathrm{B}_{n\text{-}1} \swarrow \mathrm{C}_{n\text{-}1} \downarrow\downarrow \mathrm{A}_{n\text{-}1} \searrow \mathrm{B}_{n\text{-}1}$$

$$\mathrm{C}_n : \mathrm{C}_{n\text{-}1} \nwarrow \mathrm{D}_{n\text{-}1} \leftarrow\leftarrow \mathrm{B}_{n\text{-}1} \swarrow \mathrm{C}_{n\text{-}1}$$

$$\mathrm{D}_n : \mathrm{D}_{n\text{-}1} \nearrow \mathrm{A}_{n\text{-}1} \uparrow\uparrow \mathrm{C}_{n\text{-}1} \nwarrow \mathrm{D}_{n\text{-}1}$$

Draw S_2, appropriately labeling the subpieces that are visible.

3. Find a closed-form solution for the recurrence relation defined by

$$a_n = a_{n-1} + 12a_{n-2} \quad \text{for } n \geq 2, \qquad \text{and } a_0 = 0, a_1 = 14.$$

4. Find a closed-form solution for the recurrence relation defined by

$$a_n = 2a_{n-1} - 3 \quad \text{for } n \geq 1, \qquad \text{and } a_0 = 1.$$

5. A recursive divide-and-conquer algorithm has a complexity function that satisfies $f(1) = 4$ and $f(n) = 8f\left(\frac{n}{4}\right) + 5n$. Find a good big-$\Theta$ reference function for f.

6. The following is a correct algorithm for searching an unsorted list, a, of integers for the integer, x. If x is present, it returns the first position in the list where x occurs; otherwise it returns -1. The algorithm assumes that $n = 2^k$, for some $k \in \mathbb{N}$.

```
1:  integer
      search(x, {a0, a1, a2, ..., a(n-1)})
2:     if n == 1
3:        if a0 == x
4:           return 0
5:        else
6:           return -1
7:
8:     # search the left and right
        halves of the list
9:
10:    i = search(x, {a0, ..., a(floor(n/2)-1)})
11:    j = search(x, {a(floor(n/2)), ..., a(n-1)})
12:
13:    if i > -1
14:       return i
15:    else
16:       return j
17: end search
```

(a) What is the recurrence relation that counts the number of comparisons for this algorithm? (The critical steps are at lines 2, 3, and 13).

(b) What is a good big-Θ reference function for algorithm **search**.

7. Evaluate $\binom{-2}{4}$.

8. Use generating functions to find a closed-form solution for the following recurrence relation.

- $a_n = 2a_{n-1} + 5$
- $a_0 = 1$

9. The generating function for $\frac{1}{1+z}$ is $1 - z + z^2 - z^3 + \cdots$. Calculate the generating function for $\frac{1}{(1+z)^2}$.

7.4 Projects

Mathematics

1. Find out what a Sierpinksi gasket is. Write a short expository paper.

2. Write a paper about linear nonhomogeneous recurrence relations with constant coefficients. Present a solution technique and provide a few instructive examples.

3. Find a version of the master theorem that is appropriate for recurrence relations of the form

$$a_1 f\left(\left\lfloor \frac{n}{b} \right\rfloor\right) + a_2 f\left(\left\lceil \frac{n}{b} \right\rceil\right) + cn^v.$$

Provide a proof for the theorem.

4. Write a brief expository paper about exponential generating functions.

5. Obtain a copy of *Concrete Mathematics* by Graham, Knuth, and Patashnik [43]. Master the solution of the original Josephus problem (with every third person eliminated). Write a self-contained, coherent exposition of the solution.

Computer Science

1. Write a program to produce Persian rugs.

2. Write a program to draw Sierpinski curves. The program should animate the process. Make the drawing slow enough to see the curves as they evolve.

3. Write a program to use adaptive quadrature to do numerical integration.

4. Write a program to simulate the Towers of Hanoi.

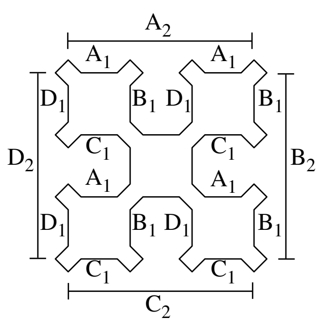
5. Write a program to simulate the Josephus problem (with every third person eliminated).

7.5 Solutions to Sample Exam Questions

1. This is the same as algorithm `PascalTriangle`, with a few changes in notation.

```
integer Cnr(integer n, integer r)
  if (n == 0) or (r == n)
     return 1
  else
     return Cnr(n-1,r-1) +
            Cnr(n-1,r)
end Cnr
```

2. The following diagram shows S_2, with the recursions (excluding the base case) annotated.



3. This is a linear homogeneous recurrence relation with constant coefficients. The characteristic equation is

$$x^2 - x - 12 = (x+3)(x-4) = 0.$$

The roots are $r_1 = -3$ and $r_2 = 4$. The general solution is therefore

$$a_n = \theta_1(-3)^n + \theta_2(4)^n.$$

The coefficients can be determined by solving the following linear system of equations.

$$\begin{aligned} \theta_1 + \theta_2 &= 0 \\ -3\theta_1 + 4\theta_2 &= 14 \end{aligned}$$

The first equation implies that $\theta_1 = -\theta_2$. Substituting into the second equation leads to $7\theta_2 = 14$, so $\theta_2 = 2$ and $\theta_1 = -2$.

Thus,

$$a_n = (-2)(-3)^n + 2(4)^n \quad \text{for — } n \geq 0.$$

A quick check is in order. The following table compares a few values, calculated first directly from the recurrence relation, and second from the closed-form formula.

n	via recurrence relation	$(-2)(-3)^n + 2(4)^n$
0	0	0
1	14	14
2	14	14
3	182	182
4	350	350

4. This is not homogeneous. Back substitution is the simplest way to find a closed-form solution for this recurrence relation.

$$\begin{aligned} a_n &= 2a_{n-1} - 3 \\ &= 2\left[2a_{n-2} - 3\right] - 3 && \text{substitute} \\ &= 2^2 a_{n-2} - 3\cdot 2 - 3\cdot 2^0 && \text{simplify} \\ &= 2^2\left[2a_{n-3} - 3\right] - 3\cdot 2 - 3\cdot 2^0 && \text{substitute} \\ &= 2^3 a_{n-3} - 3\cdot 2^2 - 3\cdot 2 - 3\cdot 2^0 && \text{simplify} \\ &\ \vdots && \vdots \\ &= 2^n a_0 - 3\sum_{i=0}^{n-1} 2^i \\ &= 2^n + (-3)\frac{2^n - 1}{2 - 1} \\ &= 2^n + (-3)(2^n - 1) \\ &= -2^{n+1} + 3 \end{aligned}$$

Thus, $a_n = -2^{n+1} + 3$, for $n \geq 0$. This can be checked against the original recurrence relation for a few values.

n	via recurrence relation	$-2^{n+1} + 3$
0	1	1
1	−1	−1
2	−5	−5
3	−13	−13
4	−29	−29

5. The master theorem (version 2) with $a = 8$, $b = 4$, $c = 5$, $d = 4$, and $v = 1$ applies. Since $a = 8 > 4^1 = b^v$, and $\log_4(8) = \frac{3}{2}$ $(4 \cdot \sqrt{4} = 8)$, $f \in \Theta(n\sqrt{n})$.

6. (a) The base case uses two comparisons, so $d = 2$. All other invocations also use two comparisons (lines 2 and 13), so $c = 2$. There are two recursive invocations, each on a list half the size of the original, so $a = b = 2$. The recurrence relation for this algorithm (with respect to comparisons) is $f(1) = 2$ and $f(n) = 2f\left(\frac{n}{2}\right) + 2$.

(b) Since $a > 1$, version 1 of the master theorem implies that $f \in \Theta(n)$.

7. $\displaystyle\binom{-2}{4} = \frac{(-2)(-3)(-4)(-5)}{4!} = \frac{120}{24} = 5$

8. Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function for the recurrence relation. Then

$$\sum_{n=1}^{\infty} a_n z^n = 2\sum_{n=1}^{\infty} a_{n-1} z^n + 5\sum_{n=1}^{\infty} z^n.$$

Thus

$$A(z) - a_0 = 2z\sum_{k=0}^{\infty} a_k z^k + 5\left(\frac{1}{1-z} - 1\right).$$

This simplifies to

$$A(z)(1-2z) = 5 \cdot \frac{1}{1-z} - 4,$$

so

$$A(z) = 5 \cdot \frac{1}{1-2z} \cdot \frac{1}{1-z} - 4\frac{1}{1-2z}.$$

Table 7.10 implies

$$A(z) = 5 \cdot \left(\sum_{k=0}^{\infty} 2^k z^k\right)\left(\sum_{k=0}^{\infty} z^k\right) - 4\sum_{k=0}^{\infty} 2^k z^k.$$

Theorem 7.22 can be used to change this to

$$\begin{aligned} A(z) &= 5\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} 2^j 1^{k-j}\right) z^k - 4\sum_{k=0}^{\infty} 2^k z^k \\ &= \sum_{k=0}^{\infty}\left(5 \cdot \frac{2^{k+1}-1}{2-1} - 4 \cdot 2^k\right) z^k \\ &= \sum_{k=0}^{\infty}\left(3 \cdot 2^{k+1} - 5\right) z^k. \end{aligned}$$

In summary,

$$A(z) = \sum_{n=0}^{\infty}\left(3 \cdot 2^{n+1} - 5\right) z^n.$$

Thus, $a_n = 3 \cdot 2^{n+1} - 5$, for $n \geq 0$.

This can be checked against the original recurrence relation for a few values.

n	$a_n = 2a_{n-1} + 5$	$3 \cdot 2^{n+1} - 5$
0	1	1
1	7	7
2	19	19
3	43	43
4	91	91

9. Use Theorem 7.22.

$$\frac{1}{(1+z)^2} = \frac{1}{1+z} \cdot \frac{1}{1+z} = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^j(-1)^{k-j}\right) z^k$$

It is appropriate at this point to stop and determine the value of the inner summation for a few values of k. The following table helps to organize this information.

k	0	1	2	3
$\sum_{j=0}^{k}(-1)^j(-1)^{k-j}$	1	-2	3	-4

There seems to be a very simple pattern. In fact, it is easy to see how the pattern arises. When k is even, $k - j$ will be even whenever j is even and odd whenever j is odd. Thus, the product, $(-1)^j(-1)^{k-j}$, will be 1 whenever k is even. The sum will add $k+1$ one's. On the other hand, if k is odd, $k - j$ will be odd whenever j is even and even when j is odd. Thus, the product, $(-1)^j(-1)^{k-j}$, will be -1 whenever k is even. The sum will add $k+1$ one's.

This can be summarized very simply:

$$\sum_{j=0}^{k}(-1)^j(-1)^{k-j} = (-1)^k(k+1).$$

It is now possible to complete the original task.

$$\begin{aligned} \frac{1}{(1+z)^2} &= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}(-1)^j(-1)^{k-j}\right) z^k \\ &= \sum_{k=0}^{\infty}(-1)^k(k+1)z^k \\ &= 1 - 2z + 3z^2 - 4z^3 + 5z^4 - \cdots. \end{aligned}$$

This can also be done by using derivatives. Start with

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$$

and take the first derivative of both sides. The result is

$$-\frac{1}{(1+z)^2} = 0 - 1 + 2z - 3z^2 + \cdots.$$

Consequently,

$$\frac{1}{(1+z)^2} = 1 - 2z + 3z^2 - \cdots.$$