CHAPTER

8

Review

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8.71 **The $(j, k; m)$ Ramsey Condition**

8.72 **$R(j, k; m)$**

8.2 Occupancy Problems

Solutions to all eight categories of occupancy problems, where n represents the number of objects and k represents the number of containers.

		Containers: Distinguishable		Containers: Indistinguishable	
Objects	Distinguishable	$\emptyset$:	k^n	$\emptyset$:	$\sum_{i=1}^{k} S(n,i)$
		$\neg\emptyset$:	$k!S(n,k)$	$\neg\emptyset$:	$S(n,k)$
	Indistinguishable	$\emptyset$:	$\binom{k+n-1}{n}$	$\emptyset$:	$\sum_{i=1}^{k} p(n,i)$
		$\neg\emptyset$:	$\binom{n-1}{k-1}$	$\neg\emptyset$:	$p(n,k)$

$\emptyset$: containers may be empty

$\neg\emptyset$: containers must contain at least one object

8.3 Sample Exam Questions

1. A small high school is holding its annual awards banquet. There are 15 awards (such as "most improved student," "best in math") that will be distributed among the nine graduating seniors. A student may receive more than one award, but the school authorities make sure that every student receives at least one award. In how many ways can the awards be distributed?

2. List all the partitions of 7.

3. Orthogonal Latin squares

(a) What is the maximum number of mutually orthogonal Latin squares of order n that are possible?

(b) How many mutually orthogonal Latin squares of order 6 are possible?

(c) Briefly describe the connection between mutually orthogonal Latin squares of order n and finite projective planes of order n.

4. Let $\mathcal{F}$ be a finite projective plane. Suppose that there is a line in $\mathcal{F}$ that contains $n+1$ points.

(a) What is the order of $\mathcal{F}$?

(b) How many lines contain each point in $\mathcal{F}$?

(c) How many points are in $\mathcal{F}$?

5. BIBDs

(a) Produce a $(13, 4, 1)$-design by starting with the block $\begin{matrix}0\\1\\3\\9\end{matrix}$ and adding 1 (mod 13).

(b) Recall the basic necessary conditions for the existence of a balanced incomplete block design with parameters (v, b, r, k, λ):

$$bk = vr$$

and

$$r(k-1) = \lambda(v-1).$$

Prove the first condition: $bk = vr$.

(c) (Extra credit) Prove the second condition: $r(k-1) = \lambda(v-1)$.

6. A knapsack has volume 8. Use algorithm **Knapsack** to find an optimal packing, given the three items described by the following table.

Item	X	Y	Z
Benefit	4	6	3
Volume	3	4	2
Quantity	3	1	2

7. Recall the encoding scheme for a 7-bit Hamming code.

$$x_5 = x_2 + x_3 + x_4 \pmod 2$$
$$x_6 = x_1 + x_3 + x_4 \pmod 2$$
$$x_7 = x_1 + x_2 + x_4 \pmod 2$$

The string 1101000 has been received.

(a) What is the nearest code word?

(b) What is the best estimate of the message that was sent?

(c) What is the Hamming weight of the received string?

8. How are the minimum distance in a linear code and Hamming weight related?

9. Systems of distinct representatives

(a) Define *system of distinct representatives*.

(b) Find all systems of distinct representatives for the sets $A_1 = \{2, 3\}$, $A_2 = \{2, 3\}$, and $A_3 = \{1, 3\}$.

(c) State a condition that characterizes when a collection of sets has a system of distinct representatives.

10. Ramsey numbers

(a) Define the Ramsey number $R(j, k)$.

(b) Prove that $R(4, 2) = 4$.

8.4 Projects

Mathematics

1. Write a brief expository paper on R. A. Fisher's use of combinatorial designs in his work on the design of (statistical) experiments.
2. Write a brief expository paper about the Catalan numbers. Include a discussion of their role in counting.
3. Write a brief expository paper about *derangements*.
4. Write a brief expository paper about the Hamming bound for error-correcting codes.
5. Write a brief expository paper about Hadamard designs and Hadamard matrices.
6. Write a report that contains a proof of Ramsey's theorem (Theorem 8.75).

Computer Science

1. Write a program to calculate $p(n, k)$ and $p(n)$.
2. Write a program to calculate $S(n, k)$ and $s(n, k)$.
3. Write a program that implements algorithm **Knapsack**. There should be an option to print (or suppress printing) the table of intermediate calculations.
4. Find out how to encode and decode messages in a 15-bit Hamming code. Then write a program that implements these operations.
5. Write a program which finds a set of $n - 1$ mutually orthogonal Latin squares of order n. Make a crude estimate of a big-Θ reference function for your algorithm before deciding how large n can be and still have the program terminate in a reasonable amount of time.
6. Write a program that determines a system of distinct representatives for a collection of sets. Use the (inefficient) algorithm described just after Example 8.51 on page 507.

8.5 Solutions to Sample Exam Questions

1. This is an **OD CD ¬∅** container problem with $n = 15, k = 9$. There are consequently $9!S(15, 9)$ ways to distribute the awards. [In case you are curious, $9!S(15, 9) = 24{,}359{,}586{,}451{,}200$.]

2. There are $p(7) = 15$ distinct partitions of 7:

$7 = 7$	$7 = 6 + 1$
$7 = 5 + 1 + 1$	$7 = 4 + 3$
$7 = 4 + 1 + 1 + 1$	$7 = 3 + 3 + 1$
$7 = 3 + 2 + 1 + 1$	$7 = 3 + 1 + 1 + 1 + 1$
$7 = 2 + 2 + 1 + 1 + 1$	$7 = 2 + 1 + 1 + 1 + 1 + 1$

$7 = 5 + 2$
$7 = 4 + 2 + 1$
$7 = 3 + 2 + 2$
$7 = 2 + 2 + 2 + 1$
$7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$

3. (a) Theorem 8.21 implies that there can be at most $n - 1$ mutually orthogonal Latin squares of order n.

(b) It is not possible to find even two mutually orthogonal Latin squares of order 6.

(c) Corollary 8.31 states the following:
A finite projective plane of order n exists if and only if there is a set of $n - 1$ mutually orthogonal Latin squares of order n.

4. (a) A finite projective plane with $n + 1$ points per line has order n (Definition 8.27).

(b) Each point is contained by $n + 1$ lines (Theorem 8.26).

(c) There are $n^2 + n + 1$ points in $\mathcal{F}$ (Theorem 8.26).

5. (a) The blocks can be generated as in the following table.

0	1	2	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11	12	0
3	4	5	6	7	8	9	10	11	12	0	1	2
9	10	11	12	0	1	2	3	4	5	6	7	8

It is traditional to rearrange the blocks as

0	1	2	3	0	1	2	3	4	5	0	1	0
1	2	3	4	4	5	6	7	8	9	6	7	2
3	4	5	6	5	6	7	8	9	10	10	11	8
9	10	11	12	7	8	9	10	11	12	11	12	12

(b) See Theorem 8.35.
The first equation is verified by counting all the 1s in M two different ways.
Since there are b blocks, each containing k varieties, each of the b columns of M will contain k 1s, for a total of bk 1s. On the other hand, each of the v varieties is in r blocks, so each of the v rows of M contains r 1s, for a total of vr 1s. Therefore, $bk = vr$.

(c) See Theorem 8.35.
The second equation is verified by counting the 1s in a sub-matrix of M. Start by choosing any variety, u. Delete the row of M that corresponds to u and delete every column that corresponds to a block that does not contain u. Now count the 1s in the matrix, M_u, that remains.
Since u is in r blocks, there will be r columns in M_u. Each of those columns will contain $k - 1$ 1s (since the 1 in u's row has been removed). On the other hand, u is in λ common blocks with each of the $v - 1$ other varieties. So each of those varieties contributes λ 1s to M_u. Consequently, $r(k - 1) = \lambda(v - 1)$.

6.

	T				K		
v	X	Y	Z	$B(v)$	X	Y	Z
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	0	0	$3 + B(0) = 3$	3	0	0	1
3	$4 + B(0) = 4$	0	$3 + B(1) = 3$	4	1	0	0
4	$4 + B(1) = 4$	$6 + B(0) = 6$	$3 + B(2) = 6$	6	0	1	0
5	$4 + B(2) = 7$	$6 + B(1) = 6$	$3 + B(3) = 7$	7	1	0	1
6	$4 + B(3) = 8$	$6 + B(2) = 9$	$3 + B(4) = 9$	9	0	1	1
7	$4 + B(4) = 10$	$6 + B(3) = 10$	$3 + B(5) = 10$	10	1	1	0
8	$4 + B(5) = 11$	$B(7) = \overline{10}$	$3 + B(6) = 12$	12	0	1	2

The optimal solution is to pack one item Y and two item Z's, for a total benefit of 12. Note the change in row 8 to handle the limitation that only one item Y is available.

7. (a) The recalculated check bits are $x_5 = 0, x_6 = 0, x_7 = 1$. An error has occurred, since the received x_7 and computed x_7 differ. If x_1 were in error, both x_6 and x_7 would differ, so x_1 must be correct. Similarly, if x_2 or x_3 were in error, there would be two check bits that differ. If x_4 were in error, all three check bits would differ. Therefore, the error is in the check bit, x_7. The nearest code word is therefore 1101001.

(b) The best estimate of the original message is the first four bits of the nearest code word. For this string, the message would be 1101.

(c) $\mathrm{Hw}(1101000) = 3$

8. Corollary 8.52 states the following:

> The minimum distance, d, in a binary linear error-correcting code is the smallest nonzero Hamming weight among the code words.

9. (a) Definition 8.61:
Let $A_1, A_2, \ldots, A_n$ be n (not necessarily distinct) subsets of a set U. A list, $\{r_1, r_2, \ldots, r_n\}$, of elements in U is called a *system of distinct representatives* for $\{A_1, A_2, \ldots, A_n\}$ if

- $r_i \in A_i$, for $i = 1, 2, \ldots, n$
- $r_i \neq r_j$, for $i \neq j$

(b) There are two systems of distinct representatives: $r_1 = 2, r_2 = 3, r_3 = 1$ and $r_1 = 3, r_2 = 2, r_3 = 1$.

(c) The marriage condition:

> Let $A_1, A_2, \ldots, A_n$ be n (not necessarily distinct) subsets of a set U. The collection, $\{A_1, A_2, \ldots, A_n\}$, is said to satisfy the *marriage condition* if for every k with $1 \leq k \leq n$ and every choice of a size-k subcollection, $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$
>
> $$|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| \geq k.$$

Theorem 8.63 proves that the marriage condition does characterize the existence of a system of distinct representatives.

10. (a) Definition 8.68:

> The *Ramsey number*, $R(j, k)$, is the smallest integer such that every set, S, with at least $R(j, k)$ elements satisfies the (j, k) Ramsey condition.

This assumes Definition 8.67:
Let S be a set with n elements. Let $j \geq 2$ and $k \geq 2$. S satisfies the (j, k) *Ramsey condition* if for every partition of the two-element subsets of S into the disjoint sets, X and Y, there is either a j-element subset, T, of S such that every two-element subset of T is in X, or else there is a k-element subset, U, of S such that every two-element subset of U is in Y.

(b) See Example 8.54 on page 511.