CHAPTER

10 Review

10.1 Definitions in Chapter 10

10.1 Simple Graph—Preliminary Definition

10.2 Simple Graph

10.3 Graph

10.4 Adjacent; Incident

10.5 Degree of a Vertex; Regular

10.7 Subgraph

10.8 Complement of a Simple Graph

10.9 The Line Graph

10.10 Bipartite Graph

10.11 Clique; Independent Set

10.14 Walk; Trail; Path; Closed Walk; Circuit; Cycle

10.15 Connected

10.16 Cut Vertex; Cut Edge

10.17 Connectivity

10.18 Edge Connectivity

10.20 Adjacency Matrix of a Simple Graph

10.21 Adjacency Matrix

10.25 Euler Trail; Euler Circuit

10.28 Hamilton Path; Hamilton Cycle

10.34 Incidence Matrix

10.36 Isomorphic

10.37 Degree Sequence

10.39 Planar Graph

10.42 Elementary Subdivision of an Edge

10.43 Homeomorphic

10.45 Polyhedron

10.46 Regular Polyhedron

10.49 The Dual Graph

10.50 Chromatic Number

10.56 Degree of a Region

10.57 Simple Directed Graph

10.58 Directed Graph

10.59 Indegree; Outdegree

10.61 Adjacency Matrix of a Directed Graph

10.62 Directed Walk; Directed Circuit

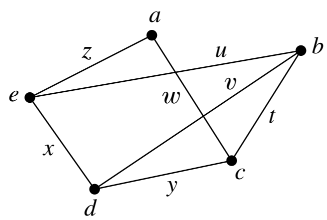
10.63 Weakly Connected

10.64 Strongly Connected

10.70 Tournament Graph;Transitive Tournament Graph

10.2 Sample Exam Questions

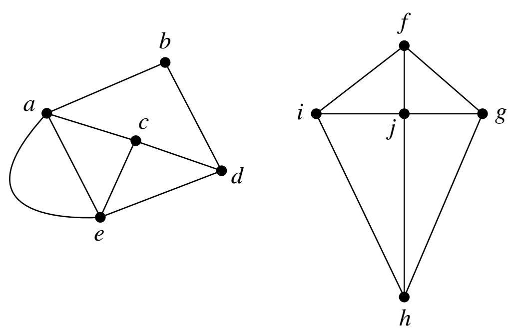
1. State, and then prove, the handshake theorem.
2. Form the complement and the line graph of C_5, the cycle graph on five vertices.
3. Consider the following graph.



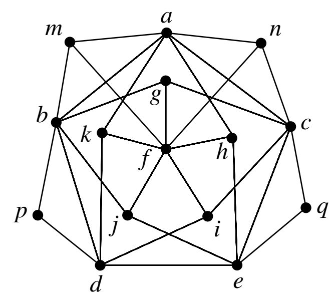
   (a) Form the adjacency matrix (labeled by the vertices in alphabetical order).
   (b) Form the incidence matrix (labeled by the vertices in alphabetical order).
   (c) What is the degree sequence for this graph?
   (d) Does this graph contain an Euler circuit or an Euler trail? If it does, list one; if it doesn't, explain why not.
4. Complete the following definition:

   A graph $G = (V, E, \phi)$ is *connected* if

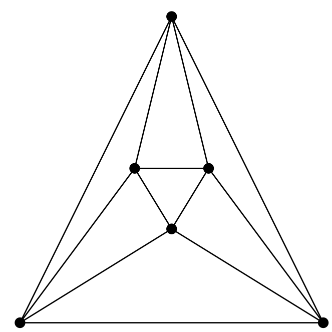
5. Are the following two graphs isomorphic? Provide sufficient justification for your answer.



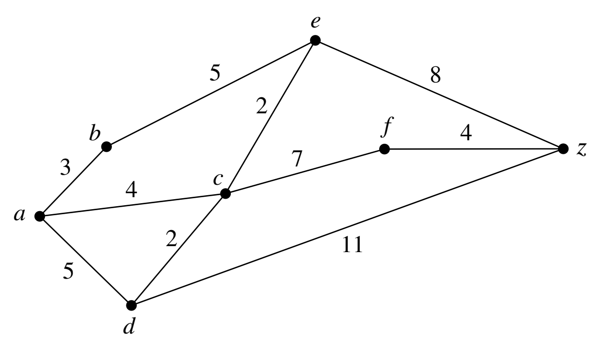
6. Is the following graph a planar graph? If so, produce a planar embedding, if not, prove that it isn't. (*Note*: I deliberately skipped over the letters l and o in the labels.)



7. List the five regular polyhedra (by name) and indicate how many edges each face is bounded by.
8. Find the chromatic number, χ, for the octahedron. Prove that your answer is correct.



9. Use Dijkstra's algorithm to find the shortest path from a to z in the following graph. List the path and its length.



10.3 Projects

Mathematics

1. Write a brief expository report on how the four-color theorem was proved. Provide sufficient background to explain the nature of the multiple cases that were examined.
2. Write a brief paper about *matchings in bipartite graphs*. Explain the connection to Hall's marriage theorem (Theorem 8.63).
3. Write a brief expository paper about *strongly regular graphs*.
4. Write a brief expository paper which explores *independent sets* and *cliques* in graphs.
5. Write a brief expository paper about the Chinese postman problem and its connection to Euler circuits.

Computer Science

1. Write a program that implements Dijkstra's shortest path algorithm.
2. Use your favorite object-oriented language to create a graph class. The class should maintain one or more useful representations of a graph, as well as a collection of useful methods.

At a minimum, there should be the ability to add and remove vertices and edges and the ability to print the adjacency and incidence matrices. It would also be useful to have a method that determines whether the graph is connected.

3. Find the definitions of *independent sets* and *cliques* in graphs. Then write a program that finds a maximal independent set and a maximal clique for any graph. Choose a representation for the graph that supports the algorithms you create.

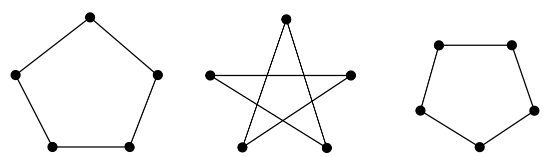
4. Write a program that takes a matrix as input and determines whether the matrix might represent the adjacency matrix of a simple graph. Make the algorithm as accurate as possible.

5. Write a program that takes the description of a graph as input and produces a diagram showing the vertices, edges, and adjacencies. The input description should contain the $x-y$ position for each vertex, as well as information about adjacencies.

10.4 Solutions to Sample Exam Questions

1. See Theorem 10.6 on page 602.

2. The relevant graphs are shown.



C_5 $\quad$ $\overline{C_5}$ $\quad$ Line Graph of C_5

3. (a) The adjacency matrix is

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

(b) The incidence matrix is

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccccc} t & u & v & w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}.$$

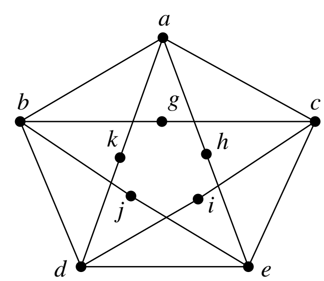
(c) 3, 3, 3, 3, 2

(d) It contains neither an Euler circuit nor an Euler trail, since there are four vertices with odd degree. (See Theorems 10.26 and 10.27.)

4. A graph $G = (V, E, \phi)$ is *connected* if there is a walk in G between every pair of vertices.

5. Their respective degree sequences are 4, 4, 3, 3, 2 and 4, 3, 3, 3, 3. Since the two degree sequences are different, the graphs are not isomorphic. (See Theorem 10.38.)

6. By removing the vertices, f, m, n, p, and q (and their incident edges), the following graph is obtained. This graph is homeomorphic to K_5, so the original graph is not planar.



7.

Polyhedron	Face shape	Bounding Edges
Tetrahedron	Equilateral triangles	3
Cube	Squares	4
Octahedron	Equilateral triangles	3
Dodecahedron	Equilateral pentagons	5
Icosahedron	Equilateral triangles	3

8. Because the graph contains triangles, at least three colors are required. It is possible to start with three colors for the central triangle. The three outer vertices are then uniquely determined. Therefore, at least three colors are required, and a coloring with three colors is possible. Consequently, the chromatic number is three.

9.

				d							p						
n	**B**	**r**	**A**	**a**	**b**	**c**	**d**	**e**	**f**	**z**	**a**	**b**	**c**	**d**	**e**	**f**	**z**
0	∅			$\boxed{0}$	∞	∞	∞	∞	∞	∞							
1	$\{a\}$	a	$\{b, c, d\}$		$\boxed{3}$	4	5					a	a	a			
2	$\{a, b\}$	b	$\{e\}$			$\boxed{4}$		8							b		
3	$\{a, b, c\}$	c	$\{d, e, f\}$				$\boxed{5}$	6	11						c	c	
4	$\{a, b, c, d\}$	d	$\{z\}$					$\boxed{6}$		16							d
5	$\{a, b, c, d, e\}$	e	$\{z\}$						$\boxed{11}$	14							e
6	$\{a, b, c, d, e, f\}$	f	$\{z\}$							$\boxed{14}$							

The shortest path (tracing backward) is z-e-c-a, so the path of length 14 is *acez*.