CHAPTER

12

Review

12.1 Definitions in Chapter 12

12.2 Sample Exam Questions

1. Let $\mathcal{R} = \{(0, 5), (0, 6), (1, 6), (2, 7), (3, 6), (4, 5)\}$ and $\mathcal{S} = \{(5, 2), (5, 4), (6, 5), (6, 8), (7, 7)\}$ (both subsets of $\mathbb{N} \times \mathbb{N}$). Determine the ordered pairs in $(\mathcal{S} \circ \mathcal{R})^{-1}$.
2. Let $\mathcal{A} = \{a, b, c, d, e\}$. Find the transitive closure of the relation $\{(a, b), (a, c), (b, d), (c, e), (d, c), (e, d)\} \subseteq \mathcal{A} \times \mathcal{A}$.
3. Every positive integer, i, can be uniquely written in the form $i = q2^j$, where q is odd and $j \geq 0$. Let $\mathcal{R}$ be the relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ defined by $(n, m) \in \mathcal{R}$ if and only if $q_1 = q_2$, where $n = q_1 2^j$, $m = q_2 2^k$, and q_1 and q_2 are both odd.
   (a) Prove that $\mathcal{R}$ is an equivalence relation.
   (b) Describe the equivalence classes.
4. Consider the following relations in a relational database. The database is kept by an apartment manager. The apartment complex requires one tenant in each apartment to be designated as the "chief tenant." All other tenants in the apartment are designated as "subsidiary tenants."

Chief

Chief Tenant	Apt	Phone	Lease Expires	Deposit
Mary Wilk	212	222-3333	5-1-2004	\$250
Will Boyd	103	222-4444	5-1-2004	\$200
Mort Daud	310	222-5555	7-1-2004	\$300
Char Kane	411	222-1111	2-1-2005	\$250

Subsidiary

Name	Apt	SPhone	Chief Tenant
Polly Boyd	103	7222-4444	Will Boyd
Joe Bertonelli	310	222-2222	Mort Daud
Peter Parks	310	222-5555	Mort Daud
Billie Kane	411	222-1111	Char Kane
Mary Kane	411	222-1111	Char Kane

   (a) Form the projection, Subsidiary[SPhone,Chief Tenant].
   (b) Form the join, Chief*Subsidiary.
   (c) Find a suitable primary key for each of the two original relations. Are there any foreign keys?
5. Define *key* (in the context of relational databases).
6. The following relation has the primary key {Student, Level}. The only other key is {Student, Audition piece}. The relation stores information about students in a piano competition.

Competitors

Student	Level	Audition piece	Teacher
Anderson	1	The Happy Rabbit	Smith
Anderson	2	Für Elise	Smith
Anderson	3	La Campanella	Jones
Juarez	1	The Happy Rabbit	Jones
Holter	1	The Happy Rabbit	Jones
Laureano	2	Für Elise	Colatti
Lin	3	La Campanella	Gao

   The essential dependencies are listed next.

   {Student, Level} → Audition piece
   {Student, Level} → Teacher
   Level → Audition piece
   Audition piece → Level

   Convert the relation to a collection of relations in third normal form.
7. Convert the binary expression, $(x + \overline{y}) \cdot \overline{(x \cdot y)} + 0$, into disjunctive normal form.
8. Use the Quine–McCluskey algorithm to minimize the binary function $f(x, y, z) = x \cdot y \cdot \overline{z} + \overline{x} \cdot y \cdot \overline{z} + \overline{x} \cdot \overline{y} \cdot z + \overline{x} \cdot \overline{y} \cdot \overline{z}$.
9. Suppose you want a combinatorial circuit which outputs a 0 whenever both its binary inputs are the same, and outputs a 1 if its inputs are different.
   (a) Produce a binary function that represents the circuit.
   (b) Design the circuit using AND, OR, and NOT gates.

12.3 Projects

Mathematics

1. Write a brief expository paper describing how to use Karnaugh maps to minimize binary expressions.
2. Write a paper that provides a more careful introduction to Boyce–Codd normal form.
3. Write a brief expository paper about partially ordered sets and lattices.
4. Write a report that explains the connection between Stirling numbers of the second kind and counting *onto functions* having finite domain and range.

Computer Science

1. Write a brief expository paper that explores the ways in which current combinatorial circuit design differs from the presentation in this book (i.e., using AND, OR, and NOT gates).
2. Write a program that takes a binary function in disjunctive normal form as input and produces a table listing the value of the function for each combination of input values.
3. Write a program that takes a finite relation as input and determines which of the following properties hold for the relation: reflexive, symmetric, transitive, antireflexive, antisymmetric, asymmetric.
4. Write a program that takes as input a pair of relations from a relational database and outputs their join. The input should also specify the attributes for each table.

12.4 Solutions to Sample Exam Questions

1. It is first necessary to determine $\mathcal{S} \circ \mathcal{R}$. Since $(0, 5) \in \mathcal{R}$ and $(5, 2)$ and $(5, 4)$ are in $\mathcal{S}$, the ordered pairs $(0, 2)$ and $(0, 4)$ are in $\mathcal{S} \circ \mathcal{R}$. Continuing in this manner, we find that

$$\mathcal{S} \circ \mathcal{R} = \{(0, 2), (0, 4), (0, 5), (0, 8), (1, 5), (1, 8), (2, 7), (3, 5), (3, 8), (4, 2), (4, 4)\}.$$

Consequently,

$$(\mathcal{S} \circ \mathcal{R})^{-1} = \{(2, 0), (4, 0), (5, 0), (8, 0), (5, 1), (8, 1), (7, 2), (5, 3), (8, 3), (2, 4), (4, 4)\}.$$

2. The first pass results in

$$\{(a, b), (a, c), (\mathbf{a}, \mathbf{d}), (\mathbf{a}, \mathbf{e}), (\mathbf{b}, \mathbf{c}), (b, d), (\mathbf{c}, \mathbf{d}), (c, e), (d, c), (\mathbf{d}, \mathbf{e}), (\mathbf{e}, \mathbf{c}), (e, d)\}.$$

A second pass produces

$$\{(a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (\mathbf{b}, \mathbf{e}), (\mathbf{c}, \mathbf{c}), (c, d), (c, e), (d, c), (\mathbf{d}, \mathbf{d}), (d, e), (e, c), (e, d), (\mathbf{e}, \mathbf{e})\}.$$

No additional ordered pairs are added in a third pass, so the transitive closure is the relation listed in the second pass.

3. (a) Let $n \in \mathbb{Z}^+$, with $n = q_1 2^i$ and $m \in \mathbb{Z}^+$, with $m = q_2 2^j$, where p and q are both odd, and $i, j \geq 0$. Then $(n, n) \in \mathcal{R}$ because $q_1 = q_1$. Also, if $(n, m) \in \mathcal{R}$, then $q_1 = q_2$. But then $q_2 = q_1$, so $(m, n) \in \mathcal{R}$ is also true. Finally, let $s \in \mathbb{Z}^+$ with $s = q_3 2^k$. If $(n, m) \in \mathcal{R}$ and $(m, s) \in \mathcal{R}$, then $q_1 = q_2$ and $q_2 = q_3$. Therefore, $q_1 = q_3$, so $(n, s) \in \mathcal{R}$ is also true.
The relation, $\mathcal{R}$, is reflexive, symmetric, and transitive. It is therefore an equivalence relation.

(b) Each equivalence class consists of an odd number and its even multiples. Thus, one equivalence class is

$$[1] = \{1, 2, 4, 8, 16, \ldots\}$$

and another is

$$[3] = \{3, 6, 12, 24, 48, \ldots\}.$$

4.

Chief

Chief Tenant	Apt	Phone	Lease Expires	Deposit
Mary Wilk	212	222-3333	5-1-2004	$250
Will Boyd	103	222-4444	5-1-2004	$200
Mort Daud	310	222-5555	7-1-2004	$300
Char Kane	411	222-1111	2-1-2005	$250

Subsidiary

Name	Apt	SPhone	Chief Tenant
Polly Boyd	103	7222-4444	Will Boyd
Joe Bertonelli	310	222-2222	Mort Daud
Peter Parks	310	222-5555	Mort Daud
Billie Kane	411	222-1111	Char Kane
Mary Kane	411	222-1111	Char Kane

(a) The projection can be formed by keeping the columns for attributes SPhone, and Chief Tenant, and then removing any duplicate rows.

Subsidiary[SPhone,Chief-Tenant]

SPhone	Chief Tenant
222-4444	Will Boyd
222-2222	Mort Daud
222-5555	Mort Daud
222-1111	Char Kane

(b) The suggested algorithm starts by forming the Cartesian product of the two relations. (Lease Expires and Deposit have been abbreviated here. The phone numbers have also been temporarily truncated.)

Chief × Subsidiary

Chief Tenant	Apt	Phone	Lease-Exp	Dep	Name	Apt	SPhone	Chief Tenant
Mary Wilk	212	3333	5-1-2004	$250	Polly Boyd	103	4444	Will Boyd
Mary Wilk	212	3333	5-1-2004	$250	Joe Bertonelli	310	2222	Mort Daud
Mary Wilk	212	3333	5-1-2004	$250	Peter Parks	310	5555	Mort Daud
Mary Wilk	212	3333	5-1-2004	$250	Billie Kane	411	1111	Char Kane
Mary Wilk	212	3333	5-1-2004	$250	Mary Kane	411	1111	Char Kane
Will Boyd	103	4444	5-1-2004	$200	Polly Boyd	103	4444	Will Boyd
Will Boyd	103	4444	5-1-2004	$200	Joe Bertonelli	310	2222	Mort Daud
Will Boyd	103	4444	5-1-2004	$200	Peter Parks	310	5555	Mort Daud
Will Boyd	103	4444	5-1-2004	$200	Billie Kane	411	1111	Char Kane
Will Boyd	103	4444	5-1-2004	$200	Mary Kane	411	1111	Char Kane
Mort Daud	310	5555	7-1-2004	$300	Polly Boyd	103	4444	Will Boyd
Mort Daud	310	5555	7-1-2004	$300	Joe Bertonelli	310	2222	Mort Daud
Mort Daud	310	5555	7-1-2004	$300	Peter Parks	310	5555	Mort Daud
Mort Daud	310	5555	7-1-2004	$300	Billie Kane	411	1111	Char Kane
Mort Daud	310	5555	7-1-2004	$300	Mary Kane	411	1111	Char Kane
Char Kane	411	1111	2-1-2005	$250	Polly Boyd	103	4444	Will Boyd
Char Kane	411	1111	2-1-2005	$250	Joe Bertonelli	310	2222	Mort Daud
Char Kane	411	1111	2-1-2005	$250	Peter Parks	310	5555	Mort Daud
Char Kane	411	1111	2-1-2005	$250	Billie Kane	411	1111	Char Kane
Char Kane	411	1111	2-1-2005	$250	Mary Kane	411	1111	Char Kane

Now remove any row for which the two versions of Chief Tenant and the two versions of Apartment (Apt) are not the same.

Reduced Chief × Subsidiary

Chief Tenant	Apt	Phone	Lease Exp	Dep	Name	Apt	SPhone	Chief Tenant
Will Boyd	103	4444	5-1-2004	$200	Polly Boyd	103	4444	Will Boyd
Mort Daud	310	5555	7-1-2004	$300	Joe Bertonelli	310	2222	Mort Daud
Mort Daud	310	5555	7-1-2004	$300	Peter Parks	310	5555	Mort Daud
Char Kane	411	1111	2-1-2005	$250	Billie Kane	411	1111	Char Kane
Char Kane	411	1111	2-1-2005	$250	Mary Kane	411	1111	Char Kane

Finally, remove one copy of each duplicate column.

Chief * Subsidiary

Chief Tenant	Apartment	Phone	Lease Expires	Deposit	Name	SPhone
Will Boyd	103	222-4444	5-1-2004	$200	Polly Boyd	222-4444
Mort Daud	310	222-5555	7-1-2004	$300	Joe Bertonelli	222-2222
Mort Daud	310	222-5555	7-1-2004	$300	Peter Parks	222-5555
Char Kane	411	222-1111	2-1-2005	$250	Billie Kane	222-1111
Char Kane	411	222-1111	2-1-2005	$250	Mary Kane	222-1111

(c) Suitable primary keys for Chief are either Chief Tenant or Apartment or Phone. The best choice is Chief Tenant, since it is used as a foreign key in the Subsidiary relation. The only choice for primary key for Subsidiary is Name.

5. This is Definition 12.25.

Let $\mathcal{T}$ be a relation and A be the attribute set of $\mathcal{T}$. A nonempty subset, $\mathsf{P} = \{A_1, A_2, \ldots, A_j\}$, of A is called a *key* for $\mathcal{T}$ if:

1. All attributes in $\mathsf{A} - \mathsf{P}$ (the set difference) are functionally dependent on P.
2. No nonempty proper subset of P has property 1.

6. The functional dependencies Level → Audition piece and Audition piece → Level indicate that this relation is not in second normal form (for example, D = Level is properly contained in the primary key).

The algorithm for converting to third normal form can begin by setting D = {Student, Level} and B = Teacher. Then replace Competitors by Competitors[Student, Level, Teacher] and Competitors[Student, Level, Audition piece].

Competitors[Student, Level, Teacher]

Student	**Level**	**Teacher**
Anderson	1	Smith
Anderson	2	Smith
Anderson	3	Jones
Juarez	1	Jones
Holter	1	Jones
Laureano	2	Colatti
Lin	3	Gao

Competitors[Student, Level, Audition piece]

Student	**Level**	**Audition piece**
Anderson	1	The Happy Rabbit
Anderson	2	Für Elise
Anderson	3	La Campanella
Juarez	1	The Happy Rabbit
Holter	1	The Happy Rabbit
Laureano	2	Für Elise
Lin	3	La Campanella

The first table is already in third normal form (the only essential dependency is {Student, Level} → Teacher).

The second relation still has the functional dependencies Level → Audition piece and Audition piece → Level, but there are no choices for B in the algorithm for which B is not in any key. Therefore, this table is also in third normal form.

It is still tempting to do one more lossless decomposition, replacing Competitors[Student, Level, Audition piece] with Competitors[Student, Level] and Competitors[Level, Audition piece]. The relation Competitors[Student, Level, Audition piece] can be regained as Competitors[Student, Level]*Competitors[Level, Audition-piece]. This would, however leave us with a table with nothing in it but the primary key.

7. Follow the process outlined in the text.

Move complements onto single variables.

$$\begin{aligned} (x+\overline{y})\cdot\overline{(x\cdot y)}+0 &= (x+\overline{y})\cdot\overline{(x\cdot y)} && \text{identity}\\ &= (x+\overline{y})\cdot(\overline{x}+\overline{y}) && \text{DeMorgan}\end{aligned}$$

Now replace this with a sum of products.

$$\begin{aligned} &(x+\overline{y})\cdot(\overline{x}+\overline{y})\\ &=(x+\overline{y})\cdot\overline{x}+(x+\overline{y})\cdot\overline{y} && \text{distributivity}\\ &=\overline{x}\cdot(x+\overline{y})+\overline{y}\cdot(x+\overline{y}) && \text{commutativity (twice)}\\ &=(\overline{x}\cdot x+\overline{x}\cdot\overline{y})+(\overline{y}\cdot x+\overline{y}\cdot\overline{y}) && \text{distributivity (twice)}\\ &=(x\cdot\overline{x}+\overline{x}\cdot\overline{y})+(x\cdot\overline{y}+\overline{y}\cdot\overline{y}) && \text{commutativity (twice)}\\ &=x\cdot\overline{x}+\overline{x}\cdot\overline{y}+x\cdot\overline{y}+\overline{y}\cdot\overline{y} && \text{associativity}\end{aligned}$$

Now make each term a minterm.

$$\begin{aligned} &x\cdot\overline{x}+\overline{x}\cdot\overline{y}+x\cdot\overline{y}+\overline{y}\cdot\overline{y}\\ &=0+\overline{x}\cdot\overline{y}+x\cdot\overline{y}+\overline{y}\cdot\overline{y} && \text{complement}\\ &=0+\overline{x}\cdot\overline{y}+x\cdot\overline{y}+\overline{y} && \text{idempotence}\\ &=\overline{x}\cdot\overline{y}+0+x\cdot\overline{y}+\overline{y} && \text{commutativity}\\ &=\overline{x}\cdot\overline{y}+x\cdot\overline{y}+\overline{y} && \text{identity}\\ &=\overline{x}\cdot\overline{y}+x\cdot\overline{y}+x\cdot\overline{y}+\overline{x}\cdot\overline{y} && \text{replacement algorithm }(x)\\ &=\overline{x}\cdot\overline{y}+(x\cdot\overline{y}+x\cdot\overline{y})+\overline{x}\cdot\overline{y} && \text{associativity}\\ &=\overline{x}\cdot\overline{y}+x\cdot\overline{y}+\overline{x}\cdot\overline{y} && \text{idempotence}\\ &=\overline{x}\cdot\overline{y}+\overline{x}\cdot\overline{y}+x\cdot\overline{y} && \text{commutativity}\\ &=(\overline{x}\cdot\overline{y}+\overline{x}\cdot\overline{y})+x\cdot\overline{y} && \text{associativity}\\ &=\overline{x}\cdot\overline{y}+x\cdot\overline{y} && \text{idempotence}\end{aligned}$$

It is easy to check that the expression in disjunctive normal form is equivalent to the original function.

x	y	$(x+\overline{y})\cdot\overline{(x\cdot y)}+0$	$\overline{x}\cdot\overline{y}+x\cdot\overline{y}$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	0	0

8. $f(x,y,z) = x\cdot y\cdot\overline{z}+\overline{x}\cdot y\cdot\overline{z}+\overline{x}\cdot\overline{y}\cdot z+\overline{x}\cdot\overline{y}\cdot\overline{z}$

Phase 1:

1	$x\cdot y\cdot\overline{z}$	110	√	1, 2	– 1 0
2	$\overline{x}\cdot y\cdot\overline{z}$	010	√	2, 4	0 – 0
3	$\overline{x}\cdot\overline{y}\cdot z$	001	√	3, 4	0 0 –
4	$\overline{x}\cdot\overline{y}\cdot\overline{z}$	000	√		

Phase 2:

	1 $x\cdot y\cdot\overline{z}$	**2** $\overline{x}\cdot y\cdot\overline{z}$	**3** $\overline{x}\cdot\overline{y}\cdot z$	**4** $\overline{x}\cdot\overline{y}\cdot\overline{z}$
– 1 0	×	×		
0 – 0		×		×
0 0 –			×	×

The most efficient way to cover all four columns is to use rows one and three. The minimized binary expression is therefore

$$\overline{x}\cdot\overline{y}+y\cdot\overline{z}.$$

9. (a)

x	y	$f(x, y)$
0	0	0
0	1	1
1	0	1
1	1	0

This translates to $f(x, y) = \overline{x} \cdot y + x \cdot \overline{y}$, which is already minimized.

(b)