Vandermonde's Identity

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1 Motivation

According to Graham, Knuth and Patashnik in their book *Concrete Mathematics*, Vandermonde's Identity is in the "Top Ten Binomial Coefficient Identities" [GKP94]. French mathematician Alexandre-Théophile Vandermonde's achievements are even more astounding when one considers that he was not interested in mathematics until the age of 35.

Encouraged to pursue music from the time of his birth in 1735, Vandermonde played the violin and eventually directed the Conservatoire des Arts et Métiers in 1782. He also belonged to the Académie des Sciences, where he submitted a work on the connection between science and music. Though he enjoyed both fields, Vandermonde argued that music theory should not be based on mathematics. Curiously, Vandermonde was opposed to the dominant Greek interpretation of music; he did not see music as a mathematical art.

Vandermonde also submitted four mathematical papers to the Academy that included the study of roots, topology, factorials, and determinants. Lastly, he provided the world of mathematics with a combinatorial identity whose simple proof has made its way into many studies of combinatorics [Uni05].

2 Preliminary Ideas

You should be familiar with the definitions of *independent* and *mutually exclusive*, and both the *Rule of Sum* and *Rule of Product* counting principles. It is also important that you understands the concept of choosing, as well as the notation for *combinations*. If these terms are unfamiliar to you, please refer to the "Common Definitions" file. That file also contains a brief review of summation notation.

3 The Problem Presented

Theorem 1 *Vandermonde's Convolution Identity*

Let m, n, and r be nonnegative integers, with $r \leq m$ and $r \leq n$. Then

$$\binom{m+n}{r} = \sum_{i=0}^{r} \binom{m}{i} \cdot \binom{n}{r-i}.$$

4 The Solution by Counting

Consider a bag of yellow and blue balls. Suppose there are m blue balls and n yellow balls. You decide to choose a collection of r balls out of this bag. How many ways can that be done? Consistent with the approach to most combinatorial proofs, we decide to start with the side of the equality that looks easiest. In this case, it is the left side. Clearly, by definition of a combination, $\binom{m+n}{r}$ represents the number of ways to choose r balls from a bag containing a total of $m+n$ balls.

Now we consider the right side of the equality. A subset of size r will have some blue balls and some yellow balls. Start by considering the case that all r balls are yellow and none are blue. Since picking a yellow ball is independent of picking a blue ball, the number of ways to complete each task is multiplied together to give a total of $\binom{m}{0} \cdot \binom{n}{r}$ ways to complete them both. Now pick one blue ball; the remaining $(r-1)$ balls are yellow. Similarly, there are $\binom{m}{1} \cdot \binom{n}{r-1}$ ways to complete both tasks. Continue in this pattern until reaching the case where r blue balls and 0 yellow balls are chosen. Since you cannot pick different numbers of yellow and blue balls simultaneously, each of these cases is mutually exclusive. Thus, the *Rule of Sum* implies that we add. We arrive at the summation $\sum_{i=0}^{r} \binom{m}{i} \cdot \binom{n}{r-i}$.

Equating these two counting methods completes the proof.

$\square$

5 Visual Example

Suppose there are two blue balls ($m = 2$) and four yellow balls ($n = 4$) in a bag. You want to pull two out ($r = 2$). How many ways can this be done?

The visualization begins by asking you to list all of these $\binom{2+4}{2}$ ways. The balls are distinguished by color, with the two blue balls labelled 1 and 2, and the four yellow balls labelled 1 through 4. After you have listed all of the ways that these pairs can be chosen, the visualization moves on to the second counting method. As in the proof, all possible pairs are sorted into cases based on how many blue balls are picked, (starting with zero blue balls and ending with two blue balls). Remember that the visualization is not itself a proof; it considers only this specific case.

References

[GKP94] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics*. Addison-Wesley, second edition, 1994.

[Gos03] Eric Gossett. *Discrete Math With Proof*. Prentice Hall, 2003.

[RM+99] Kenneth H. Rosen, John G. Michaels, et al., editors. *Handbook of Discrete and Combinatorial Mathematics*. CRC Press, 1999.

[Uni05] University of St. Andrews, http://www-groups.dcs.st-and.ac.uk/ history/. *The MacTutor History of Mathematics Archive*, May 2005.