Fibonacci Identity 2

Callie Wurtz

The **Motivation** for the Fibonacci sequence along with its **Preliminary Ideas** can be found in the "Fibonacci Overview" file. That document also contains a combinatorial interpretation of the Fibonacci numbers that you are advised to read before continuing.

1 The Problem Presented

Theorem 1

For $n \geq 0$,

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1.$$

2 The Solution by Counting

Consider tiling an $(n+2)$-board with squares and dominoes. We are concerned with how many of these tilings use at least one domino. By our interpretation of Fibonacci numbers, there are f_{n+2} ways to tile a board of length $(n+2)$. Excluding the "all square" tiling gives $f_{n+2} - 1$ tilings that include at least one domino.

Since we know there must be at least one domino, we can consider the position of the last domino. Let this last domino cover cells $k+1$ and $k+2$. (Note that if the board begins with its only domino, then $k = 0$.) This implies that cells $k+3$ through $n+2$ must be covered by squares. Therefore, the only cells we are unsure about for $k > 0$ are cells 1 through k. Consequently, there are f_k ways to tile these cells. Since the last domino cannot be in two places at once, the *Rule of Sum* applies. There are thus $f_0 + f_1 + f_2 + \cdots + f_n = \sum_{k=0}^{n} f_k$ ways to tile a board with at least one domino.

Equating the two ways we counted this number completes the proof.

□

3 Visual Example

Consider a board of length 4+2 = 6. The visualization first lists every way to tile this board. Note that there are $f_6 = 13$ ways to do this. Without the "all-square" tiling, there are $f_{4+2} - 1 = 12$ tilings. Now for the left-hand side, consider the position of the last domino. For example, if the last (and only) domino is the first piece on the board, it covers cells 1 and 2. This implies that the rest of the pieces are squares, and thus, there is only $f_0 = 1$ way that this arrangement can be tiled. The visualization breaks down the remaining summands from the left-hand side and demonstrates that they add to 12.

References

[BQ03] Arthur T. Benjamin and Jennifer J. Quinn. *Proofs that Really Count: The Art of Combinatorial Proof*. Number 27 in The Dolciani Mathematical Expositions. The Mathematical Association of America, 2003.