The Tulip Problem

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1 Motivation

This identity is located in the "Combinatorial Proof" problem section of Dr. Eric Gossett's *Discrete Mathematics With Proof* textbook. In the discrete mathematics class at Bethel University the "Tulip Problem" and its clever solution have become synonymous with combinatorial proof. When asked what they remember about combinatorial proofs, many former discrete students answered, "I remember counting tulips."

2 Preliminary Ideas

You should be familiar with the definition of *mutually exclusive*, the notation for *combinations*, and the *Rule of Sum* counting principle. If these terms are unfamiliar to you, please refer to the "Common Definitions" file. That file also contains a review of summation notation and its properties.

3 The Problem Presented

Lemma 1

Let n and k be nonnegative integers. Then

$$\sum_{b=0}^{k} \binom{n+(k-b)}{k-b} = \binom{n+k+1}{k}.$$

4 The Solution by Counting

Consider a row of tulips in a garden. A gardener wants to plant red and blue tulips. She is interested in the number of visually distinct patterns. Since there are only two colors, these patterns can be determined by the blue tulips.

Let there be $n+1$ red tulips and k blue tulips. Then the number of ways to organize these tulips can be found by choosing the position of the blue tulips from all of the possible positions. This number is given by $\binom{(n+1)+k}{k}$.

Consider this organization in another way. Looking at the garden row from left to right, we see that one way to distinguish patterns is based on which tulips begin the pattern. Let b be the number of blue tulips in the row before we hit a red tulip. Notice that the set determined by a particular value of b is mutually exclusive from every other set determined by a different value of b, (implying the *Rule of Sum*). If there are b blue tulips and then a red tulip, we have determined the position of 1 of the $n+1$ red tulips and b of the k blue tulips. Thus, there are $((n+1)-1)+(k-b)$ tulips left to arrange, with $k-b$ of them being blue. There are $\binom{n+(k-b)}{k-b}$ patterns with exactly b blue tulips before the first red tulip is encountered. In conclusion, there are $\sum_{b=0}^{k}\binom{n+(k-b)}{k-b}$ distinct patterns.

Equating these two ways of counting the gardener's choices gives $\sum_{b=0}^{k}\binom{n+(k-b)}{k-b} = \binom{n+k+1}{k}$, thus proving the lemma.

$\square$

Though the sum on the left-hand side of the lemma provides a good combinatorial interpretation, the identity is usually simplified and presented in the following form.

Theorem 1

Let n and k be nonnegative integers. Then

$$\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}.$$

Proof:

The sum $\sum_{b=0}^{k} \binom{n+(k-b)}{k-b}$ can be simplified to $\sum_{j=0}^{k} \binom{n+j}{j}$ by noticing that

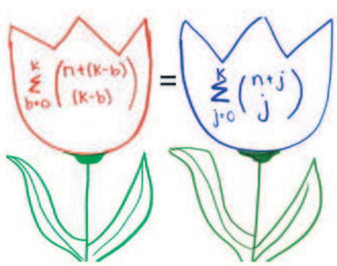
$$\sum_{b=0}^{k} \binom{n+(k-b)}{k-b} = \binom{n+k}{k} + \binom{n+(k-1)}{k-1} + \cdots + \binom{n+1}{1} + \binom{n}{0}$$

and

$$\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+(k-1)}{k-1} + \binom{n+k}{k}.$$

Therefore, $\sum_{b=0}^{k} \binom{n+(k-b)}{k-b} = \sum_{j=0}^{k} \binom{n+j}{j}$. By Lemma 1, $\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k}$.

□



A visual representing the simplification of the sums.[1]

5 Visual Example

Suppose Barb has two red tulip bulbs ($n = 1$) and three blue tulip bulbs ($k = 3$). She is interested in the number of visually distinct ways she can plant these in one row. The visualization begins with the right-hand side of the identity and lists all $\binom{1+3+1}{3} = 10$ ways to plant these flowers and then highlights them according to the left-hand side of the lemma: $\sum_{b=0}^{3} \binom{1+(3-b)}{3-b}$. As in the proof, this example begins with the case when $b = 0$ (the row starts with a red tulip). Keep in mind that though the visualization follows the proof, this example is not a proof in and of itself.

References

[Gos03] Eric Gossett. *Discrete Math With Proof.* Prentice Hall, 2003.

[1]The drawing, inspired by the tulip problem, was originally created in poster-size for Dr. Gossett's discrete math class by students Kelly Kirkwood and Liz O'Connor.