Pascal's Triangle

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1 Motivation

Pascal's triangle was around long before Blaise Pascal worked on it in the 17th Century. Known as the arithmetic triangle, the date of its inception is a mystery.

It is significant to note that 13th Century Chinese mathematician Yang Hui was one of the first to reference the arithmetic triangle [Kat93, p. 192]. Based on Hui's work, it is presumed that this triangle was used in Asian culture even before his time [BM93, p. 231]. However, Pascal's *Treatise on the Arithmetical Triangle* was the most prominent Western work on the topic. Consequently, it is Pascal's name that is associated with this special triangle as well as many of its properties. Though Pascal's triangle has been used for centuries, the numeric relationships found within this triangle continue to fascinate people.

2 Preliminary Ideas

Definition 1 *Pascal's Triangle*

Pascal's triangle is a number triangle with numbers arranged in staggered rows. Columns are determined relative to each row rather than relative to the whole table. The numbering of the columns and rows begins at 0. The entry in row n and column r of the triangle contains the binomial coefficient $\binom{n}{r}$.

The first 6 rows of Pascal's triangle are shown below. Notice that the entry in row 4 and column 2 is 6. This means that $\binom{4}{2}$ should equal 6, which it does.

Row 0						1					
Row 1					1		1				
Row 2				1		2		1			
Row 3			1		3		3		1		
Row 4		1		4		6		4		1	
Row 5	1		5		10		10		5		1

Figure 1: Pascal's triangle

You should also be familiar with the definition of *mutually exclusive*, the notation for *combinations*, and the *Rule of Sum* counting principle. If these terms are unfamiliar to you, please refer to the "Common Definitions" file.

3 The Problem Presented

Theorem 1 *Pascal's Theorem*

Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

4 The Solution by Counting

Consider a class of n people. In this class, we want to choose a committee of size r. Clearly, by definition of a combination, there are $\binom{n}{r}$ ways to do this.

Now, focus on a specific person in the class. We'll call her Kelly. Either Kelly is on the committee or she is not. In fact, all of the possible committees of size r can be partitioned into two disjoint groups: those that contain Kelly and those that do not. Thus, the *Rule of Sum* counting principle applies to this problem. If Kelly is not on the committee, there are still r committee spots to fill but now only $n-1$ people to choose from, which gives $\binom{n-1}{r}$ choices. If Kelly is on the committee, there is now one less committee member that needs to be chosen in order to get a committee of size r. Thus, there are $\binom{n-1}{r-1}$ choices. Therefore, there are a total of $\binom{n-1}{r} + \binom{n-1}{r-1}$ ways to choose a committee of size r from a group of size n.

Equating these two counting results completes the proof.

$\square$

5 Visual Example

Consider a basketball team of 7 players. Denote the players by the initials MJ, BR, KAJ, LB, PM, DW, and OR. How many ways can the coach choose 5 players for his starting line-up?

The visualization goes through the proof as it relates to this particular example. (Remember that the visualization itself is not a proof.) The illustration begins with the left-hand side and lists all of the possible starting line-ups for these 7 players. As expected, there are $\binom{7}{5} = 21$ ways to create this line-up.

The next part of the visualization corresponds to the right-hand side. The same number of starting line-ups is counted based on the mutually exclusive options of whether or not PM (Pete Maravich[1]) starts. Note that there are $\binom{6}{5} = 6$ ways for "Pistol Pete" to start and $\binom{6}{4} = 15$ ways for him not to start.

6 Using Pascal's Triangle

This example can also be visualized using Pascal's triangle. Recall that in Pascal's triangle, the combination of $\binom{7}{5} = 21$ is the entry in the 7th row and 5th column. Find this entry.

Now look to the previous row: row 6. Find the numbers that correspond to the 5th and 4th columns in this row. These numbers, $\binom{6}{5} = 6$ and $\binom{6}{4} = 15$, sum to 21.

[1] See *www.PistolPete23.com* for information about the late Pete Maravich and his legacy.

References

[BM93] Carl B. Boyer and Uta C. Merzbach. *A History of Mathematics*. John Wiley & Sons, Inc., 1993.

[Gos03] Eric Gossett. *Discrete Math With Proof*. Prentice Hall, 2003.

[Kat93] Victor J. Katz. *History of Mathematics: An Introduction*. HarperCollins College Publishers, 1993.

[RM+99] Kenneth H. Rosen, John G. Michaels, et al., editors. *Handbook of Discrete and Combinatorial Mathematics*. CRC Press, 1999.