BIBD

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1 Motivation

Consider this famous problem first posed by Thomas Kirkman in 1850.
In a boarding school there are fifteen schoolgirls who always take their daily walks in rows of threes. The headmistress desires that no two schoolgirls walk in the same row more than once a week. The arrangement of the schoolgirls uses a combinatorial design known as a Balanced Incomplete Block Design. The use of these designs however extends beyond solving classic mathematical problems. BIBD's are used in many areas including coding theory, cryptography, group testing, and even tournament scheduling [RM$^+$99, p. 760].

2 Preliminary Ideas

Definition 1 *Balanced Incomplete Block Design*

The *Balanced Incomplete Block Design*, abbreviated *BIBD*, is a type of combinatorial design that consists of a finite collection of finite sets, called *blocks*. These sets each contain a finite collection of elements, called *varieties*. The BIBD follows specific properties that are expressed by the positive integer parameters (v, b, r, k, λ). They are

v: the number of varieties
b: the number of blocks
r: the number of blocks to which each variety belongs
k: the number of varieties in each block
λ: the number of blocks in which every pair of distinct varieties appears.

Because the blocks have a uniform size, and each variety appears in the same number of blocks, and each pair of varieties appears in the same number of blocks, the design is said to be *balanced*. The term *incomplete* refers to the fact that not every variety is present in every block.

A balanced incomplete block design can be efficiently displayed as a matrix.

Definition 2 *The Incidence Matrix of a BIBD*

The *incidence matrix of a BIBD* is the $v \times b$ matrix $M = (m_{ij})$ defined by

$$m_{ij} = \begin{cases} 1 & \text{if the } i\text{th variety is in the } j\text{th block;} \\ 0 & \text{otherwise.} \end{cases}$$

3 The Problem Presented

Theorem 1 *The Parameters of a BIBD*

In a balanced incomplete block design with parameters (ν, b, r, k, λ),

$$bk = \nu r$$

and

$$r(k-1) = \lambda(\nu - 1).$$

4 The Solution by Counting

Consider M, the incidence matrix for the (ν, b, r, k, λ)-design. Both equations will be proved combinatorially by counting 1's in M in two different ways.

For the first equation, recall that there are b blocks that each contain k varieties. By definition of the incidence matrix, each of the b columns contains k 1's, for a total of bk 1's in M. Now recall that each of the ν varieties is contained in r blocks. In other words, each of the ν rows of M contains r 1's, for a total of νr 1's in M. Setting the number of 1's in M equal gives us $bk = \nu r$.

To prove the second equation, we consider a submatrix of M. Choose any particular variety, ν_i. Delete the row of M that corresponds to ν_i, and delete all of the columns that correspond to a block that does **not** contain ν_i. We will count the 1's in this remaining submatrix of M. Since ν_i is in r blocks, there will be r columns in the submatrix. Because the 1's in the row of ν_i have been deleted, each of these r columns contain $(k-1)$ 1's. Thus, there are $r(k-1)$ 1's in this submatrix of M.

Now consider counting 1's in another way. Recall that only blocks containing the variety ν_i are left in this submatrix. By the definition of λ, the variety ν_i is still in λ common blocks with each of the $\nu-1$ other varieties. So, there are λ 1's in each of the remaining $\nu-1$ rows. Consequently, there are $\lambda(\nu-1)$ 1's in the submatrix of M. The number of 1's in this submatrix is $r(k-1) = \lambda(\nu-1)$, which is the desired equality.

□

5 Visual Example

The visualization walks through the proof as it applies to the following example. Keep in mind that the visualization itself is not a proof; it simply illustrates the ideas of the proof through a particular example.

Consider this $(7, 14, 6, 3, 2)$-design.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}
0	0	0	0	0	0	1	1	1	1	2	2	2	2
1	1	3	3	5	5	3	3	4	4	3	3	4	4
2	2	4	4	6	6	5	6	5	6	5	6	5	6

The incidence matrix (with column and row labels) for this example is as follows:

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}
v_0	1	1	1	1	1	1	0	0	0	0	0	0	0	0
v_1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
v_2	1	1	0	0	0	0	0	0	0	0	1	1	1	1
v_3	0	0	1	1	0	0	1	1	0	0	1	1	0	0
v_4	0	0	1	1	0	0	0	0	1	1	0	0	1	1
v_5	0	0	0	0	1	1	1	0	1	0	1	0	1	0
v_6	0	0	0	0	1	1	0	1	0	1	0	1	0	1

Convince yourself of the parameters before stepping through the proof. Note that there are 7 rows ($v = 7$) and 14 ($b = 14$) columns. There are 6 ones in each row ($r = 6$) and 3 ones in each column ($k = 3$). Lastly, for any two varieties, they both have 1's in exactly 2 of the same columns ($\lambda = 2$).

Note that in the second part of the proof, the variety that we have chosen to delete is v_4. The remaining submatrix is 6×6. (Make sure you understand where this dimension comes from.)

References

[Gos03] Eric Gossett. *Discrete Math With Proof.* Prentice Hall, 2003.

[RM+99] Kenneth H. Rosen, John G. Michaels, et al., editors. *Handbook of Discrete and Combinatorial Mathematics.* CRC Press, 1999.