A Summation Identity

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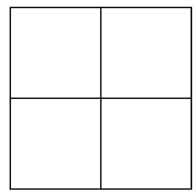
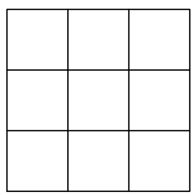
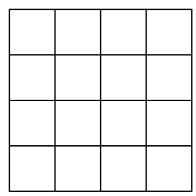
1 Motivation

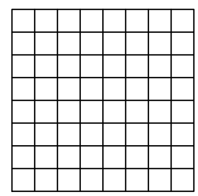
How many rectangles can you find in this 2×2 square?



In a 3×3 square? In a 4×4 square? How about an 8×8 square?







The generalized answer to these questions demonstrates an interesting connection between cubes, triangular numbers, and squares. Robert G. Stein used this question as the inspiration for a combinatorial proof that the sum of the first n cubed numbers is equal to the square of the nth triangular number [Ste71, p. 161-2]. Though the identity is often proved using mathematical induction, Stein notes that the induction proof is "quite unenlightening." Counting the number of rectangles in an $n \times n$ square in two different ways gives a clever combinatorial proof.

2 Preliminary Ideas

You should be familiar with the terms *independent* and *mutually exclusive*; the *Rule of Sum* and the *Rule of Product* counting principles are implicitly used. If you are not familiar with these definitions, please refer to the "Common Definitions" file.

The proof also assumes knowledge of summation notation and its properties, including the *Change of Variable* technique. Consult the "Common Definitions" file if you are not comfortable with summation notation.

Familiarity with the following theorem is necessary.

Theorem 1 *Sum of the first* n *positive integers*

For all positive integers n,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

As you may have already predicted from the motivation section, this proof uses a few basic concepts from geometry. However, an elementary understanding of these terms is all that is necessary, so most of the definitions are given informally.

Definition 1 *Triangular Numbers*

For any positive integer n, the nth *triangular number* is given by $T_n = 1 + 2 + \cdots + (n-1) + n$. The first few triangular numbers are 1, 3, 6, 10, 15, 21, ...

The name triangular number comes from the fact that these numbers can be expressed by a regular triangle of equally spaced points [Wei05].

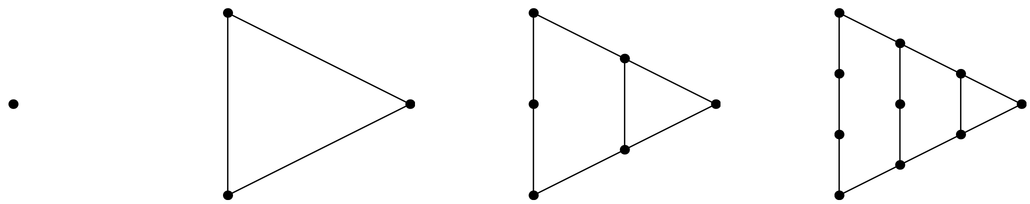


Figure 1: The triangular numbers 1, 3, 6, and 10.

Since the proof counts rectangles positioned on a square grid, it is therefore assumed that the rectangles have integer dimensions as defined below.

Definition 2 *Dimension of a Rectangle*

Let i and j be positive integers. The *dimension of a rectangle* is denoted by $i \times j$, where i corresponds to the number of unit cells in the horizontal direction and j corresponds to the number of unit cells in the vertical direction.

Definition 3 *Transformation*

A *transformation* is a one-to-one, onto mapping whose domain and range are the points in a plane.

In this exposition, the object of the transformations will be a rectangle.

Definition 4 *Translation*

A *translation* is a type of transformation. It is the rigid motion in which all points of the figure are moved in the same direction and the same distance.

Definition 5 *Rotation*

A *rotation* is also a type of transformation. One point of the plane, called the *center of rotation*, is held fixed, and the figure is turned about the center of rotation a fixed number of degrees.

3 The Problem Presented

Theorem 2

For $n > 0$,

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2.$$

4 The Solution by Counting

Note: The first counting method has been changed slightly from Stein's proof.

4.1 First Counting Method

Consider an $n \times n$ square that is divided as a grid into unit squares. We are counting the number of rectangles (recall that a square is a rectangle) in this $n \times n$ square.

The first counting method considers the vertical and horizontal movement of the rectangles. We begin by considering the case that the rectangle is a square. For an $i \times i$ square, where $1 \leq i \leq n$, there are $n - i$ spots that the square can be translated horizontally to. Including the position that it is in, there are a total of $n - i + 1$ horizontal positions for the square. In the same way, there are $n - i + 1$ vertical positions. Since its vertical movement is independent of its horizontal movement, there are a total of $(n - i + 1)^2$ squares of dimension $i \times i$ in the larger $n \times n$ square.

Now suppose the rectangle is not a square. Then it has dimensions $i \times j$, where $i \neq j$. Assume $i < j$. In the horizontal dimension, there are $(n - i + 1)$ positions for this rectangle. There are likewise $(n - j + 1)$ vertical positions for this rectangle, which are independent of the horizontal positions. Since $i \neq j$, a $90°$ rotation produces another non-square rectangle (of dimension $j \times i$). Thus there are $2 \cdot (n - i + 1) \cdot (n - j + 1)$ non-square rectangles with dimensions i and j. We now consider the choices for dimension j. Since j is strictly greater than i and different choices for j are mutually exclusive, there are $\sum_{j=i+1}^{n} 2\,(n - i + 1)(n - j + 1)$ non-square rectangles with a shorter side of dimension i.

Because a rectangle is either a square or it is not, there are a total of $(n - i + 1)^2 + \sum_{j=i+1}^{n} 2\,(n - i + 1)(n - j + 1)$ rectangles, where i is the dimension of the shorter side. All that we have left to consider are the possibilities for the dimension i. As we have already established, $1 \leq i \leq n$. Consequently, there are a total of

$$\sum_{i=1}^{n} \left[(n - i + 1)^2 + \sum_{j=i+1}^{n} 2\,(n - i + 1)(n - j + 1) \right]$$

rectangles in an $n \times n$ square. Some algebra is necessary to simplify this sum. Though it looks complicated, it is easily understood if you follow it carefully step-by-step. Begin with the inside summation and work toward the outside.

$$
\begin{aligned}
& \sum_{i=1}^{n}\left[(n-i+1)^2+\sum_{j=i+1}^{n}2\,(n-i+1)(n-j+1)\right] \\
= & \sum_{i=1}^{n}\left[(n-i+1)^2+2\,(n-i+1)\sum_{j=i+1}^{n}(n-j+1)\right] && \text{Factor the constants out of the inside summation.} \\
= & \sum_{i=1}^{n}\left[(n-i+1)^2+2\,(n-i+1)\sum_{m=1}^{n-i}(n-(m+i)+1)\right] && \text{Change of variable where } m=j-i. \\
= & \sum_{i=1}^{n}\left[(n-i+1)^2+2\,(n-i+1)\left(\sum_{m=1}^{n-i}(n-i+1)-\sum_{m=1}^{n-i}m\right)\right] && \text{Property of summations.} \\
= & \sum_{i=1}^{n}\left[(n-i+1)^2+2\,(n-i+1)\left((n-i)\cdot(n-i+1)-\frac{(n-i)(n-i+1)}{2}\right)\right] && \text{Theorem 1.} \\
= & \sum_{i=1}^{n}\left[(n-i+1)^2+2\,(n-i+1)\left(\frac{1}{2}(n-i)\cdot(n-i+1)\right)\right] && \text{Simplify inside the parentheses.} \\
= & \sum_{i=1}^{n}\left[(n-i+1)^2+(n-i+1)^2(n-i)\right] && \text{Simplify again.} \\
= & \sum_{i=1}^{n}(n-i+1)^2(1+n-i) && \text{Factor out an } (n-i+1) \text{ term.} \\
= & \sum_{i=1}^{n}(n-i+1)^3
\end{aligned}
$$

Further simplification reveals that

$$
\begin{aligned}
\sum_{i=1}^{n}(n-i+1)^3 &= (n-1+1)^3+(n-2+1)^3+\cdots+(n-(n-1)+1)^3+(n-n+1)^3 \\
&= n^3+(n-1)^3+\cdots+2^3+1^3 \\
&= \sum_{k=1}^{n}k^3.
\end{aligned}
$$

4.2 Second Counting Method

Now we will count the rectangles in an $n \times n$ square based on the location of a given rectangle's lower left vertex. Picture again the $n \times n$ square. There are n^2 places for the location of a rectangle's bottom left corner. These include every point in the square where lines meet except the upper and right-hand boundaries.

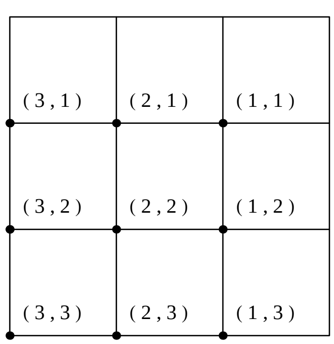


Figure 2: Example for a 3×3 square.

These points are labelled (p, q), where p is the horizontal distance from the upper right-hand corner of the square and q is the vertical distance from this corner. (Both of these distances are positive though we are moving left and down.) Note that $1 \leq p, q \leq n$.

At any given point (p, q), how many rectangles have their bottom left corner at that position? There are pq such rectangles. One can see this by noting that there are p options for the height of such a rectangle and q options for its width, and these choices are independent. Since every rectangle has a lower left vertex at only one place, the choices for p and q are mutually exclusive. Thus, there are

$$\sum_{p=1}^{n} \sum_{q=1}^{n} pq$$

rectangles in this $n \times n$ square. Again, this sum needs a little bit of simplifying.

$$\begin{aligned}
\sum_{p=1}^{n} \left(\sum_{q=1}^{n} pq \right) &= \sum_{p=1}^{n} (p \cdot 1 + p \cdot 2 + \cdots + p \cdot n) \\
&= \sum_{p=1}^{n} p \cdot (1 + 2 + \cdots + n) \\
&= 1 \cdot (1 + 2 + \cdots + n) + 2 \cdot (1 + 2 + \cdots + n) + \cdots + n \cdot (1 + 2 + \cdots + n) \\
&= (1 + 2 + \cdots + n) \cdot (1 + 2 + \cdots + n) \\
&= (1 + 2 + \cdots + n)^2 \\
&= \left(\sum_{k=0}^{n} k \right)^2
\end{aligned}$$

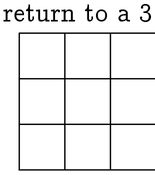
We have counted the same number of rectangles in two different ways. In doing so we have arrived at our equality:

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2 .$$

□

5 Visual Example

For the visual example, we return to a 3×3 square divided into units.



There are $1^3 + 2^3 + 3^3 = (1 + 2 + 3)^2 = 36$ rectangles in this square.

5.1 First Counting Method

For the first counting method, we will count the rectangles based on the summation

$$\sum_{i=1}^{n} \left[(n - i + 1)^2 + \sum_{j=i+1}^{n} 2\,(n - i + 1)(n - j + 1) \right] .$$

The proof counts rectangles by their smallest sides. For example, consider the case when the smallest side is 1 $(i = 1)$.

There are $(3-1+1)^2 = 9$ squares of dimension 1×1.

There are $2 \cdot (3-1+1)(3-2+1) = 2(3)(2) = 12$ rectangles with dimensions of 1 and 2.

Lastly, there are $2 \cdot (3-1+1)(3-3+1) = 2(3)(1) = 6$ rectangles with dimensions of 1 and 3.
Thus, we have a total of $9 + (12+6) = 27$ rectangles with a smaller side of 1.

5.2 Second Counting Method

As in the proof, the second counting method uses the summation

$$\sum_{p=1}^{n} \sum_{q=1}^{n} pq.$$

The rectangles are highlighted according to the position of their bottom left vertices. The visualization begins by displaying each of the rectangles that has a lower left corner at point $(p,q) = (1,1)$. We move down to point $(1,2)$, and so on. Consider the point $(3,1)$ as an example. There are $3 \cdot 1 = 3$ rectangles with a lower left corner at this point: one 3×1 rectangle, one 2×1 rectangle, and one 1×1 rectangle.

References

[Ste71] Robert G. Stein. A combinatorial proof that $\sum k^3 = (\sum k)^2$. *Mathematics Magazine*, 44(3):161–162, May 1971.

[Wei05] Eric Weisstein. *Mathworld*. Wolfram Research, http://mathworld.worlfram.com/, 2005.