Fibonacci Identity 3

The **Motivation** for the Fibonacci sequence along with its **Preliminary Ideas** can be found in the "Fibonacci Overview" file. That document also contains a combinatorial interpretation of the Fibonacci numbers that you are advised to read before continuing.

1 The Problem Presented

Theorem 1

For $m, n \geq 1$,

$$f_{m+n} = f_m f_n + f_{m-1} f_{n-1}.$$

2 The Solution by Counting

Consider tiling a board of length $m + n$ with squares and dominoes. By our interpretation of the Fibonacci numbers, there are clearly f_{m+n} ways to tile an $(m + n)$-board.

Now consider cell m. We are interested in the breakability of cell m. If cell m is breakable, then by definition, the tiling can be decomposed into an m-tiling followed by an n-tiling. Since there are f_m ways to tile an m-board and f_n ways to tile an n-board and these tasks are independent of each other, there are $f_m \cdot f_n$ ways to tile an $(m + n)$-board breakable at cell m.

Suppose now that the board is not breakable at cell m. This implies that cells m and $m + 1$ are covered by a domino. Therefore, the tiling must be breakable at cell $m - 1$ and there are f_{m-1} ways to tile cells 1 through $m-1$. Note that the board is also breakable at cell $m+1$, and there are $(m+n)-(m+2)+1 = n-1$ cells from $m + 2$ through $m + n$. Those $n - 1$ cells can be tiled in f_{n-1} ways. Thus, there are $f_{m-1} \cdot f_{n-1}$ ways to tile a board that is unbreakable at cell m. This can be seen in the following picture.

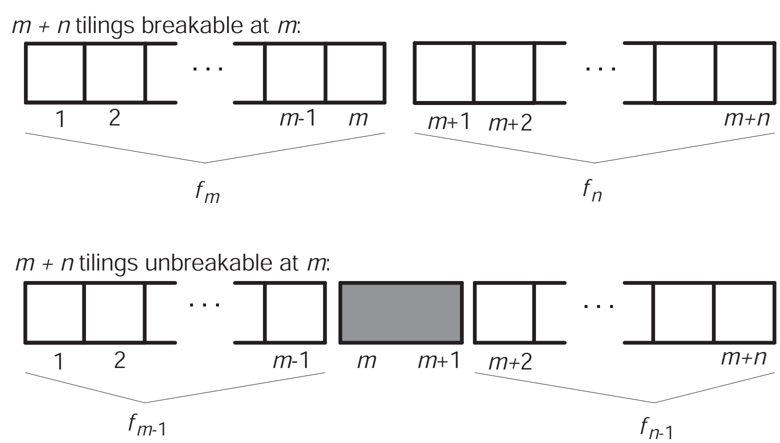

Figure 1: Counting tilings of an $(m + n)$-board based on the breakability at m.

Since the board is either breakable or unbreakable at cell m, there are a total of $f_m f_n + f_{m-1} f_{n-1}$ ways to tile a board of length $m + n$.

Equating the two ways we counted this number completes the proof.

□

3 Visual Example

Consider a board of length 6, with $m = 4$ and $n = 2$. Recall that there are $f_6 = 13$ ways to tile this board. The visualization lists these on the left side of the screen. Now consider the condition of breakability at cell 4. The visualization sorts these tilings into the two cases of the right-hand side. If it is breakable at cell 4, there are $f_4 \cdot f_2 = 5 \cdot 2$ tilings. If it is unbreakable at cell 4, there are $f_3 \cdot f_1 = 3 \cdot 1$ tilings.

References

[BQ03] Arthur T. Benjamin and Jennifer J. Quinn. *Proofs that Really Count: The Art of Combinatorial Proof*. Number 27 in The Dolciani Mathematical Expositions. The Mathematical Association of America, 2003.