Fibonacci Overview

Callie Wurtz

1 Motivation

Consider the following:
Suppose a newly-born pair of rabbits: one male, one female, are put in a field. These rabbits are able to mate at the age of one month and they give birth to a male-female pair in the following month. In other words, two months after a pair of rabbits is born, another pair is produced. Assuming no rabbits die, how many pairs will there be in one year?

This problem was first posed by Leonardo of Pisa in his work, *Liber Abaci* ("The Book of the Abacus"), which was published in 1202. Leonardo Pisano is better known as Fibonacci, but this nickname didn't appear until the 19th Century. The name Fibonacci is a shortening of *filius Bonacci*, which means "son of Bonaccio" (like the common English surname John-son). Though he never referred to himself as Fibonacci, it is this name that is attached to the famous number sequence that answers the rabbit problem [Kno05].

Fibonacci numbers are widely used in many different fields of mathematics. Countless books, websites, and even a journal, *Fibonacci Quarterly*, are devoted entirely to the study of Fibonacci numbers [Uni05].

Not surprisingly, this famous sequence of numbers is the foundation for multiple identities. While these identities are often proved using mathematical induction, Fibonacci numbers have an elegant combinatorial interpretation that allows for counting proofs.

2 Preliminary Ideas

2.1 Common Definitions

For each of the Fibonacci identities, it is assumed that you are familiar with the definitions in the "Common Definitions" file. Multiple identities implicitly use the *Rule of Sum* and *Rule of Product* counting principles as well as the terms *mutually exclusive* and *independent*. Summation notation is used in Fibonacci Identities 1, 2, and 4. Familiarity with *binomial coefficients* and the *floor function* is also necessary for "Fibonacci Identity 1."

2.2 Fibonacci Numbers Defined

Definition 1 *Fibonacci Numbers*

The *Fibonacci numbers* are defined recursively by $f_0 = 1, f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. The initial numbers of the Fibonacci sequence are $1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

The Fibonacci sequence is one famous example of a *recurrence relation*. In general, a recurrence relation is a formula that expresses a sequence of numbers, where each number in the sequence is defined in terms of one or more previous numbers in the sequence. In the case of the Fibonacci sequence, finding the term f_n is dependent on knowing both f_{n-1} and f_{n-2}. As you may have already noticed, it is necessary to have a starting point. Recurrence relations are defined with at least one base case in order to uniquely specify the sequence. In the Fibonacci sequence the base cases are $f_0 = 1$ and $f_1 = 1$. Consequently, the relation $f_n = f_{n-1} + f_{n-2}$ with the base conditions given in the previous sentence uniquely defines f_n for $n \geq 2$.

Return for a moment to the rabbit problem. This problem allowed Fibonacci to investigate and simplify a complex sequence. Table 1 depicts the rabbit population each month for a year [Gos03, p. 333].

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Baby Pairs	1	0	1	1	2	3	5	8	13	21	34	55	89
Mature Pairs	0	1	1	2	3	5	8	13	21	34	55	89	144
Total Pairs	1	1	2	3	5	8	13	21	34	55	89	144	233

Table 1: Fibonacci's Rabbit Population

Though the table does not look complicated, it is beneficial to consider the relationships present. Notice that after the first month, every mature pair is creating a baby pair, so the number of mature pairs in one month becomes the number of baby pairs in the next. Also notice that there is a similar correspondence between the second and third rows. After the first month, the total number of pairs in a month becomes the number of mature pairs in the next month.

We conclude, as Fibonacci did, that after one year, the field contains $f_{12} = 233$ rabbits.

2.3 A Visual Representation

For combinatorial purposes, a more visual representation of f_n is needed. This interpretation has been taken from *Proofs that Really Count* by Dr. Arthur T. Benjamin and Dr. Jennifer J. Quinn [BQ03].

Picture a checkerboard of dimension $1 \times n$. This board is composed of unit cells numbered 1 through n and is said to have length n. A *board of length* n is also referred to as an n-*board*. We will use squares (covering 1 cell) and dominoes (covering 2 cells) to tile the board.

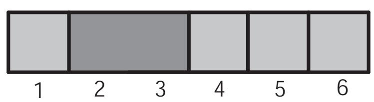

Figure 1: An arbitrary tiling of a 6-board.

Theorem 1 t_n *is a Recurrence Relation*

Let t_n denote the number of distinct tilings of a board of length n using squares and dominoes. Then t_n is a recurrence relation.

Proof:

Consider first a board of length 0. Out of convention, we say that there is one way to tile a board with no cells: it is to do nothing. In other words, $t_0 = 1$. Now consider a 1-board. Clearly, the only way to tile this board is with one square. So, $t_1 = 1$. We move on to a 2-board. This board can be tiled with 2 squares or 1 domino, giving a total of 2 tilings.

The base cases: $t_1 = 1$ and $t_2 = 2$

Suppose we have a board of length n for $n > 2$. Note that any tiling of a board of length n must begin with either a square or a domino. If it begins with a square, we are left with a board of length $n-1$, which can be tiled in t_{n-1} ways by definition. If the n-board begins with a domino, the rest of the board can be tiled in t_{n-2} ways. Since the board begins with either a square or a domino, these two options are mutually exclusive. Thus, $t_n = t_{n-1} + t_{n-2}$.

□

Corollary 1 t_n *is identical to* f_n

The number of ways to tile a board of length n with squares and dominoes is f_n.

Proof:

As can be seen in the proof of Theorem 1, the base cases for t_n are $t_0 = 1$ and $t_1 = 1$. (It was also explicitly shown that $t_2 = 2$, though only two base cases are really necessary.) The recurrence relation was given as $t_n = t_{n-1} + t_{n-2}$. One can readily see that this relation is the same as that of the Fibonacci sequence. Since the Fibonacci sequence is defined to have the same base cases and the same recurrence relation as t_n, we can conclude that the sequence given by t_n is identical to the sequence given by f_n.

$\square$

Based on the above corollary, we can replace the t_n notation with standard Fibonacci notation. All of the Fibonacci identities, written in terms of f_n, will be proved combinatorially using this visual interpretation.

The following example further illustrates the idea of our visual interpretation.

Example 1

Consider a board of length 4. There are $f_4 = 5$ ways to tile this board. They are:

Figure 2: The five tilings of a 4-board using squares and dominoes.

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Before using this combinatorial representation for Fibonacci numbers, it is necessary to introduce a few more definitions that relate to tiling boards.

Definition 2 *Breakable*

A tiling of an n-board is *breakable* at cell k if the tiling can be decomposed into two tilings: one covering cells 1 through k and the other covering cells $k+1$ through n.

This definition can be clearly seen in the following figure.

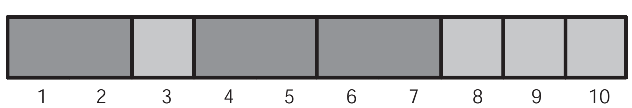

Figure 3: A 10-board tiling using 3 dominoes and 4 squares.

This board of length 10 is breakable at 2, 3, 5, 7, 8, 9, and 10. Notice that a tiling over a board of length n is always breakable at n.

The definition of unbreakable is as expected.

Definition 3 *Unbreakable*

A tiling is *unbreakable* at cell k if a domino occupies cells k and $k+1$.

The tiling in Figure 3 is unbreakable at cells 1, 4, and 6.

2.4 Pairs of Tilings

Definition 4 *Fault*

Given a board of length n placed above a board of length m, we say that there is a *fault* at cell i, where $1 \leq i \leq n$ and $1 \leq i \leq m$, if both tilings are breakable in that place. In the case that the first cells of the tilings are aligned, there is a *fault* at "cell 0."

Consider the following placement of a 10-board tiling and an 8-board tiling:

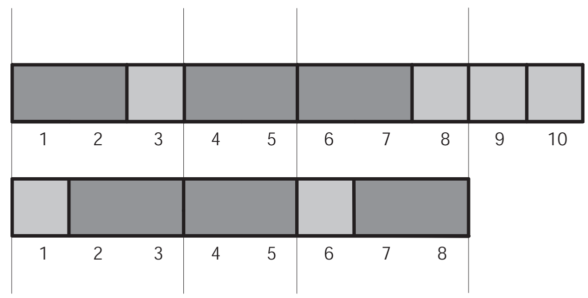

Figure 4: This pair of tilings has 4 faults.

This pair of tilings in this particular placement has 4 faults, which are denoted by the thin lines. Since both boards begin at the same place, there is a fault at cell 0 by definition. There are also faults at cells 3, 5, and 8. It is worth remembering that faults can exist at both the beginning and the end of a board (as in the case of this 8-board).

3 The Problem Presented

Each identity for the Fibonacci sequence, along with **The Solution by Counting** and corresponding **Visual Example**, is in a separate file labelled "Fibonacci Identity (#)." Though the identities can be viewed in any order, they are numbered according to my perception of their difficulty level.

References

[BQ03] Arthur T. Benjamin and Jennifer J. Quinn. *Proofs that Really Count: The Art of Combinatorial Proof.* Number 27 in The Dolciani Mathematical Expositions. The Mathematical Association of America, 2003.

[Gos03] Eric Gossett. *Discrete Math With Proof.* Prentice Hall, 2003.

[Kno05] Ron Knott. *Fibonacci Numbers and the Golden Section.* Surrey University, http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/, May 2005.

[Uni05] University of St. Andrews, http://www-groups.dcs.st-and.ac.uk/ history/. *The MacTutor History of Mathematics Archive*, May 2005.