# Fibonacci Identity 1

Callie Wurtz

The **Motivation** for the Fibonacci sequence along with its **Preliminary Ideas** can be found in the "Fibonacci Overview" file. That document also contains a combinatorial interpretation of the Fibonacci numbers that you are advised to read before continuing.

## 1 The Problem Presented

**Theorem 1**

$$\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} = f_n$$

If the notation on the left-hand side of this identity is unfamiliar to you, please consult the "Common Definitions" file.

## 2 The Solution by Counting

From the right-hand side, we already know that we are counting the number of tilings of an $n$-board. By our interpretation of the Fibonacci numbers, this number is $f_n$.

Now we are left to show that $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$ also counts this number. Notice that we can determine the tilings of a board of length $n$ by the number of dominoes that the tiling contains. Suppose a tiling contains $i$ dominoes. Because a domino takes up two cells, $0 \leq i \leq \frac{n}{2}$. Also, since there are $i$ dominoes for $n$ cells, the tiling must contain $n-2i$ squares. Thus, this tiling has a total of $i+(n-2i) = n-i$ tiles. How many ways are there to choose where the $i$ dominoes go within the $n-i$ tiles? There are $\binom{n-i}{i}$ ways. Since one tiling of an $n$-board cannot have two different numbers of dominoes, this collection of tasks is mutually exclusive. Thus, the number of tilings on an $n$-board is $\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$.

Equating the two ways we counted this number completes the proof.

□

## 3 Visual Example

Consider a board of length 5. Recall that there are $f_5 = 8$ ways to tile this board. The visualization lists each of these ways on the right side and then highlights them according to the number of dominoes that each tiling contains. As an example, note that there are $\binom{5-2}{2} = 3$ ways to tile a 5-board with 2 dominoes.

## 4 Fibonacci Numbers in Pascal's Triangle

Because this Fibonacci identity contains a binomial coefficient, it seems intuitive that it could be seen in Pascal's triangle.[1] Take a minute to study the triangle with this identity in mind. Scroll down slowly and look only at the first triangle. Can you find the Fibonacci sequence?

[1] For an introduction to the arithmetic triangle (Pascal's triangle), see the exposition entitled "Pascal's Identity."

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Figure 1: Pascal's Triangle

By summing the "shallow diagonals" the Fibonacci sequence is revealed [Wei05].

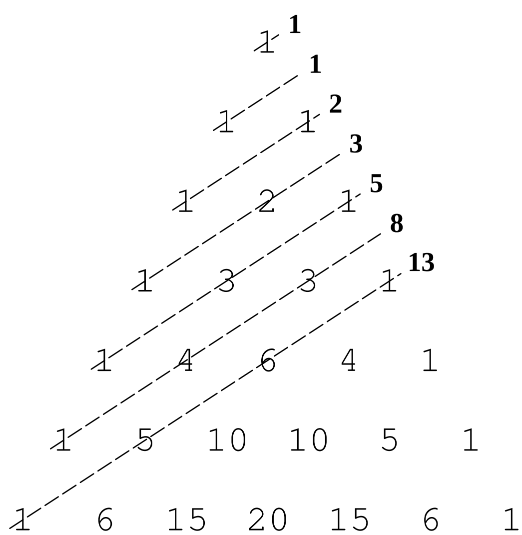


Figure 2: The Fibonacci Sequence within Pascal's Triangle.

To see how this fact relates to the above Fibonacci identity, reconsider the Fibonacci number $f_5 = 8$. By Theorem 1, we know that $f_5 = \sum_{i=0}^{\lfloor \frac{5}{2} \rfloor} \binom{5-i}{i} = \binom{5-0}{0} + \binom{5-1}{1} + \binom{5-2}{2}$. In other words, $8 = 1 + 4 + 3$, which is what we notice in the triangle. Because both the Fibonacci sequence and Pascal's triangle are such unique structures, this relationship is rather surprising.

## References

[BQ03] Arthur T. Benjamin and Jennifer J. Quinn. *Proofs that Really Count: The Art of Combinatorial Proof.* Number 27 in The Dolciani Mathematical Expositions. The Mathematical Association of America, 2003.

[Wei05] Eric Weisstein. *Mathworld.* Wolfram Research, http://mathworld.worlfram.com/, 2005.