Common Definitions

The majority of these definitions come from *Discrete Mathematics with Proof* by Dr. Eric Gossett [Gos03].

1 Combinations*

Definition 1 *Combinations*

Combinations involve choosing subsets from a given set. The order of the elements that compose the subsets is unimportant and repetition is not allowed.

Counting Formula 1 *Combinations*

The number of ways to choose a subset of r objects from a set of n objects without repetition, with $0 \leq r \leq n$, is denoted by $C(n, r)$, and is read " n choose r." The number of combinations is given by

$$C(n, r) = \frac{n!}{r! \cdot (n-r)!}.$$

If $0 \leq n < r$, then $C(n, r) = 0$.

A frequently seen notation for the combination $C(n, r)$ is $\binom{n}{r}$, which is the notation used in the Binomial Theorem. Known as a binomial coefficient, $\binom{n}{r}$ is a standard notation in combinatorial identities. Though the $C(n, r)$ notation is used in some of the visualizations, the common $\binom{n}{r}$ notation is used in the expositions.

* The reader should note that in most references to counting, *permutations* are defined along with combinations. However, I have yet to find a combinatorial proof that uses a permutation, so that definition has been omitted.

2 General Counting Principles

2.1 Independent Tasks

Definition 2 *Independent*

The tasks in a collection or sequence of tasks are said to be *independent* if the outcome of any task is not influenced by the outcomes of the other tasks in the collection or sequence.

General Counting Principle 1 *Rule of Product*

If a project can be decomposed into a collection of independent tasks with n_1 ways to accomplish the first, n_2 ways to accomplish the second, ... and n_k ways to accomplish the kth task, then the project can be completed in $n_1 n_2 \cdots n_k$ ways.

Often the word "and" is a clue that the tasks are independent. If one task "and" another are completed, they should generally be multiplied.

2.2 Mutually Exclusive Tasks

Definition 3 *Mutually Exclusive*

The tasks in a collection are *mutually exclusive* if completing any one of the tasks excludes the completion of the other tasks.

General Counting Principle 2 *Rule of Sum*

If a project can be decomposed into a collection of mutually exclusive tasks with n_1 ways to accomplish the first, n_2 ways to accomplish the second, ... and n_k ways to accomplish the kth task, then the project can be completed in $n_1 + n_2 + \cdots + n_k$ ways.

Usually the word "or" is a clue that the tasks are mutually exclusive. If one task "or" another is to be completed, they should be added.

3 Summation Notation

Summation notation is not only frequently used in the field of combinatorics but in all fields of mathematics. Familiarity with summation notation and its properties is often assumed. Those properties are given here along with a few examples.

Definition 4 *Summation Notation*

$$\sum_{i=k}^{n} a_i = a_k + a_{k+1} + a_{k+2} + \cdots + a_{n-1} + a_n$$

We say that this is the sum of the numbers a_i where i goes from k to n, inclusive. The variable i in this case is called our index variable and can be changed without altering the sum.

Recall that each term in a summation is called a *summand*. The summands in the definition are the terms $a_k, a_{k+1}, a_{k+2}, \ldots, a_{n-1}$, and a_n.

A few examples are given to reinforce the definition.

Example 1

$$\sum_{i=10}^{13} 3i = 3 \cdot 10 + 3 \cdot 11 + 3 \cdot 12 + 3 \cdot 13 = 30 + 33 + 36 + 39 = 138$$

ex

Example 2

$$\sum_{m=0}^{5} \binom{5}{m} = \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 1 + 5 + 10 + 10 + 5 + 1 = 32$$

ex

Sometimes the summation may be infinite. An example of a summation that goes to ∞ is as follows:

Example 3

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

You might recall from calculus that this sum is known as the *harmonic series*.

$\boxed{\text{ex}}$

As expected, properties of the real numbers can be applied to summation notation.

Properties:

Let c be a constant. Then

1. $\sum_{i=1}^{n} c = nc$
2. $\sum_{i=k}^{n} (a_i + b_i) = \sum_{i=k}^{n} a_i + \sum_{i=k}^{n} b_i$
3. $\sum_{i=k}^{n} (a_i - b_i) = \sum_{i=k}^{n} a_i - \sum_{i=k}^{n} b_i$
4. $\sum_{i=k}^{n} c \cdot a_i = c \cdot \sum_{i=k}^{n} a_i$.

Notice that Property 4 is akin to the distributive property. Reconsider the sum given in Example 1. Using Property 4, we see that

$$\sum_{i=10}^{13} 3i = 3 \sum_{i=10}^{13} i = 3(10 + 11 + 12 + 13) = 3(46) = 138.$$

These properties are very useful in simplifying summations. Sometimes though, a bit more "manipulation" is required. One manipulation technique that is similar to a substitution in algebra is known as a *Change of variable*. An example of this is as follows:

Example 4

Suppose you have the sum

$$\sum_{i=3}^{6} i$$

and we would like to begin at 1 instead of 3. Let us choose a new index variable: j. Notice that to start at the number 1, $j = i - 2$ since 1=3-2. The sum now looks like

$$\sum_{j=1}^{4} (j + 2).$$

In expanded form

$$\sum_{j=1}^{4} j + 2 = (1 + 2) + (2 + 2) + (2 + 3) + (2 + 4) = 3 + 4 + 5 + 6 = 18,$$

which equals

$$\sum_{i=3}^{6} i = 3 + 4 + 5 + 6 = 18.$$

$\boxed{\text{ex}}$

4 Floor and Ceiling Functions

Definition 5 *Floor Function, Ceiling Function*

For all real numbers x,

the *floor function*, denoted by $\lfloor x \rfloor$, is the largest integer in the interval $(x-1, x]$

and

the *ceiling function*, denoted by $\lceil x \rceil$, is the largest integer in the interval $[x, x+1)$.

The floor function can be thought of as a truncation, while the ceiling function always rounds up. A few examples are given to illustrate these definitions.

1. $\lfloor \frac{2}{3} \rfloor = 0$ and $\lceil \frac{2}{3} \rceil = 1$
2. $\lfloor 517.2334 \rfloor = 517$ and $\lceil 517.2334 \rceil = 518$
3. $\lfloor \pi \rfloor = 3$ and $\lceil \pi \rceil = 4$
4. $\lfloor -5.1 \rfloor = -6$ and $\lceil -5.1 \rceil = -5$

References

[Gos03] Eric Gossett. *Discrete Math With Proof.* Prentice Hall, 2003.