Counting Subsets

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1 Motivation

How many subsets does a set of size n *contain?* This is a classic mathematical question involving both the *Rule of Product* and the *Rule of Sum* counting principles. Though it is not an obvious identity, its proof is a clear demonstration of the combinatorial method of double counting.

2 Preliminary Ideas

Definition 1 *Power Set*

Let S be a set. The *power set of* S, denoted by $\mathcal{P}(S)$, is the set of all subsets of S (including the empty set and S itself).

You should also be familiar with the definitions of *independent* and *mutually exclusive*, the notation for *combinations*, and both the *Rule of Sum* and *Rule of Product* counting principles. If these terms are unfamiliar to you, please refer to the "Common Definitions" file. That file also contains a brief review of summation notation.

3 The Problem Presented

Theorem 1

Let n be a nonnegative integer. Then

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

Note that both sides of this equation are counting the number of elements in the power set. The following proof will demonstrate this.

4 The Solution by Counting

This equality lends itself well to a combinatorial proof in terms of sets and subsets. To make things more concrete, picture choosing committees (the subsets) from a class (the set). This image fits well with the definition we have of a combination. We can represent choosing a committee of size r from a class of size n as $\binom{n}{r}$. Since a committee cannot have two different sizes, choosing a committee of size s and choosing a committee of size t are mutually exclusive choices. The *Rule of Sum* counting principle then implies that the total number of committees that can be chosen from a class of size n is $\sum_{r=0}^{n} \binom{n}{r}$.

Next, visualize this class of n students standing in a line. As we go down the class-list to choose a committee, there are two options for every student. Each student is either on or off the committee. Since deciding whether one student is on the committee does not influence the decision for the next student, the selection process consists of a sequence of independent choices. The *Rule of Product* counting principle says that there are a total of $\underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{n \text{ times}}$ or 2^n committees. We have counted the same thing in two ways and arrived at our equality.

$\square$

The following corollary comes out of the proof of Theorem 1.

Corollary 1 $|\mathcal{P}(S)|$

Let S be a finite set with $n \geq 0$ elements. Then $|\mathcal{P}(S)| = 2^n$.

Using this corollary and the definition of the power set, we can now answer the initial question: a set of size n contains 2^n subsets.

5 Visual Example

Suppose Nick, Shalanah, Eric, and Allison are preparing for a project in linear algebra. They want to know how many different possibilities there are for who works on the project. This can be counted in the two ways laid out in the proof.

The visualization depicts both of these ways together. The left side lists the number of ways that these four students can work on the project based on the total number of people working on it (from 0 people to 4 people). On the right side, each of the students' names appears as headings. Recall that the two options are that a student either is or is not part of the project. If a student *is* a part of the project, a highlighted dot will appear under that student's name.

References

[Gos03] Eric Gossett. *Discrete Math With Proof.* Prentice Hall, 2003.