Oresme's Sequence

Callie Wurtz

1 Motivation

Very few notable mathematicians came out of the Middle Ages. Disasters like the Black Plague and the Hundred Years War, coupled with a lack of emphasis on education, made original thinkers like Nicole Oresme rare. Oresme was born in France in 1323. In Paris, he received an Arts Degree and then a Master of Theology in 1355. In 1377, he became Bishop of Lisieux in Normandy. He is known for his contributions to religion, philosophy, economics, and science as well as mathematics [Uni05].

Oresme was among the first to work with rules of exponents and fractional exponents. He also gave the earliest known proof for the divergence of the Harmonic series. Still, his most novel idea was to pictorially represent rates of change [BM93, p. 295-99]. Long before Galileo, Barrow, Leibniz, and Newton, Oresme introduced a graphical representation for the change of velocity over time. Oresme recognized what is now common knowledge in Calculus: the area under a curve on this graph represents distance. More interested in the theoretical than the practical, Oresme considered what happens when velocities increase without bound. He began with a unit time interval in which the velocity was 1 for the first half of the time, 2 in the next quarter of time, 3 in the next eighth of time and so on. Essentially, Oresme was summing the series $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots$. He used his geometric interpretation of distance to prove that this summation is 2 [Kat93, p. 296].

It is unfortunate that no one else in the Middle Ages built upon Oresme's work; eventually much of it was lost. Succeeding mathematicians rediscovered Oresme's original ideas. Though its existence is not well-known, Nicole Oresme structured a visual counting argument that is the epitome of a combinatorial proof.

2 Preliminary Ideas

A brief review of summation notation is included in the "Common Definitions" file.

The proof itself is quite ingenious, but it can be understood with an elementary knowledge of mathematics.

3 The Problem Presented

Theorem 1

$$\sum_{i=1}^{\infty} \frac{i}{2^i} = 2$$

Numbers of the form $\frac{i}{2^i}$ are called *Oresme numbers* [Hor74, p. 267].

4 The Solution by Counting

There is no question about which side to start with for this proof. Obviously, we begin with the right hand side, which is 2. Now, we are left to show that the left side sums to 2. To help us see this, first think of the terms of the summation. We are trying to show that $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots = 2$. We shall begin presumably as Oresme did. Picture "2" as two unit squares, which are denoted by S_1 and S_2. In order to better visualize the proof we will elongate the horizonal dimension.

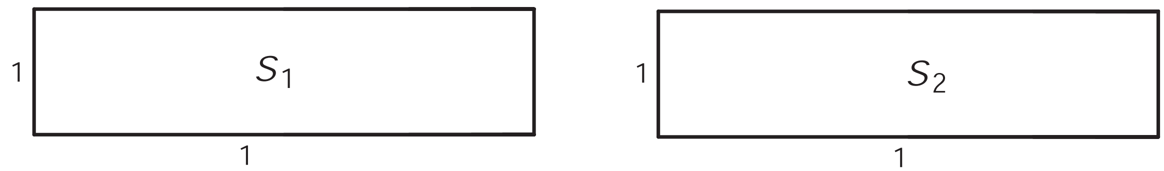

Figure 1: A visual representation of 2.

Cut each square in half vertically. The right half is then cut in half, and the rightmost fourth is cut in half. This pattern continues into infinity. As Oresme put it, the square is divided "to infinity into parts continually proportional according to the ratio 2 to 1." These *cuts* are labelled $a, b, c, \ldots$ in S_1. The *sections* (rather than the cuts) in S_2 are labelled with Greek letters. Notice that the area of α is $\frac{1}{2}$, the area of β is $\frac{1}{4}$, the area of γ is $\frac{1}{8}$, etc.

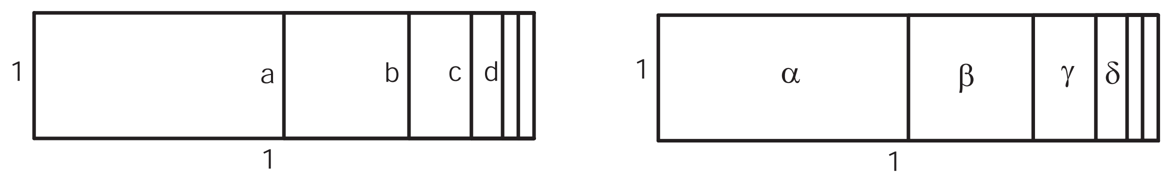

Figure 2: Both squares are halved continuously.

Using a step-ladder image, stack the pieces of S_2 onto S_1. Begin with piece α. This piece is stacked on top of the right half of S_1.

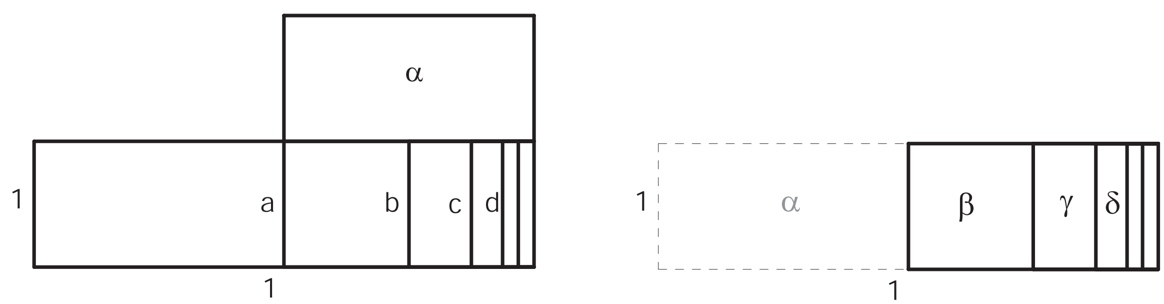

Figure 3: Block α has been moved from S_2 onto S_1.

Extend lines $b, c, d, \ldots$ through α.

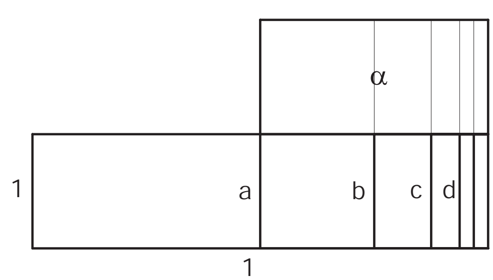

Figure 4: The lines create sections in α.

Next we take block β and stack it on top of the right half of α. Extend lines $c, d, e, \ldots$ through β. We continue this stacking to infinity, and our result is such:

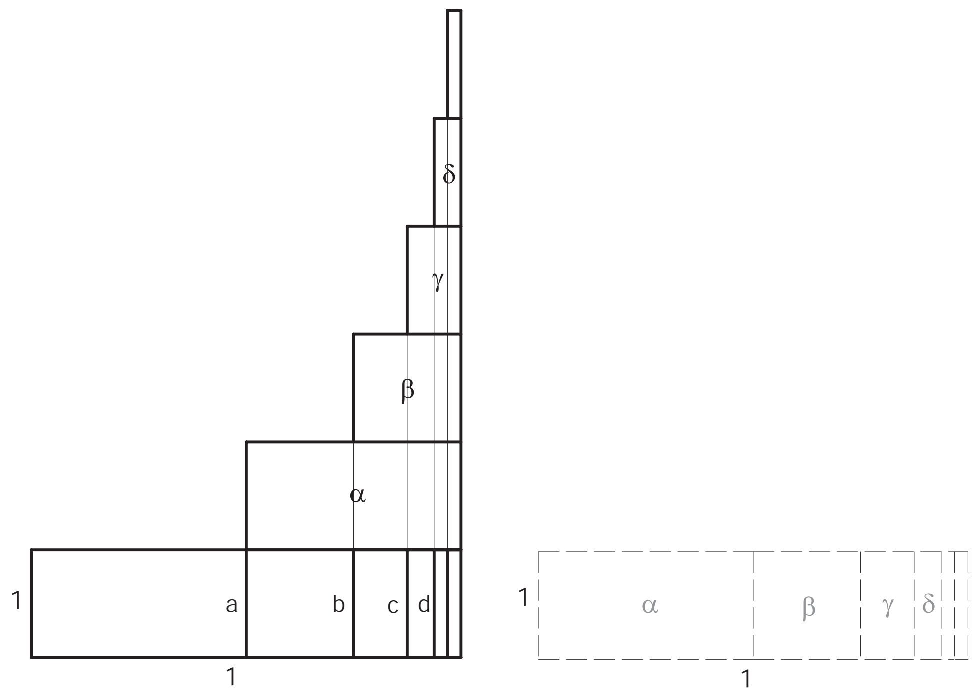

Figure 5: All of the blocks from S_2 have been stacked onto S_1.

The clever part here is in noticing what we have created. From two 1×1 squares we now have a stair-step structure representing $\frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + 5 \cdot \frac{1}{32} + \cdots$ which equals $\sum_{i=1}^{\infty} \frac{i}{2^i}$.

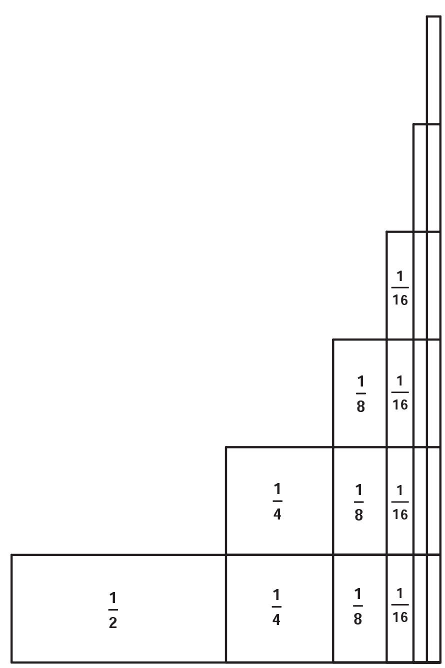

Figure 6: A visual representation of $\sum_{i=1}^{\infty} \frac{i}{2^i}$.

$\square$

The visualization further reinforces this point by beginning with the stair-step structure and moving the pieces in a tetris-like fashion to clearly reveal a 1×2 rectangle.

5 Visual Example

For this theorem, the proof itself is pictorial. No example of a specific case is necessary.

References

[BM93] Carl B. Boyer and Uta C. Merzbach. *A History of Mathematics*. John Wiley & Sons, Inc., 1993.

[Hor74] A.F. Horadam. Oresme numbers. *Fibonacci Quarterly*, 12:267–271, 1974.

[Kat93] Victor J. Katz. *History of Mathematics: An Introduction*. HarperCollins College Publishers, 1993.

[Uni05] University of St. Andrews, http://www-groups.dcs.st-and.ac.uk/ history/. *The MacTutor History of Mathematics Archive*, May 2005.