Fibonacci Identity 4

Callie Wurtz

The **Motivation** for the Fibonacci sequence along with its **Preliminary Ideas** can be found in the "Fibonacci Overview" file. That document also contains a combinatorial interpretation of the Fibonacci numbers that you are advised to read before continuing.

1 The Problem Presented

Theorem 1

For $n \geq 0$,
$$\sum_{k=0}^{n} {f_k}^2 = f_n f_{n+1}.$$

2 The Solution by Counting

Picture two domino boards, one of length n and the other of length $n+1$. We are interested in the number of ways these two boards can be tiled. By our interpretation of Fibonacci numbers, there are f_n ways to tile an n-board and f_{n+1} ways to tile an $(n+1)$-board. Since tiling one board is independent of tiling the other, there are $f_n \cdot f_{n+1}$ ways to tile both boards.

Now place the $(n+1)$-board directly above the n-board so that the first cells are aligned as in the following figure.

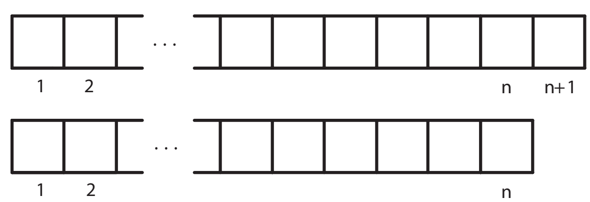

Figure 1: There are $f_n f_{n-1}$ ways to tile these boards.

We are interested in the position of the last fault. Suppose the position of the last fault is at cell k, and note that $0 \leq k \leq n$. In other words, the last and only fault may be at cell 0 or the last fault could be anywhere up to the end of the shorter board. Consider tiling the boards by first tiling to the left of the fault (the head) and then tiling to the right of the fault (the tail).

Because the last fault is at cell k, the head of each board has length k. (Recall that there is 1 way to tile a board of length 0.) There are therefore f_k ways to tile the head of the $(n+1)$-board and also f_k ways to tile the head of the n-board. Thus, there are ${f_k}^2$ ways to tile both boards through cell k.

Now consider tiling the tails of both boards. The tail of the $(n+1)$-board consists of cells $k+1$ through $n+1$. The tail of the n-board is composed of cells $k+1$ through n for $k < n$. (If $k = n$, and the tail of the n-board is length 0.) Therefore, one of the tails has an even number of cells and the other must have an odd number of cells. Because there can be no faults in the tails, there is only way to tile them. The even-length tail must be tiled with all dominoes and the odd-length tail must begin with a square and then be tiled with all dominoes. Convince yourself that this is the only possibility.

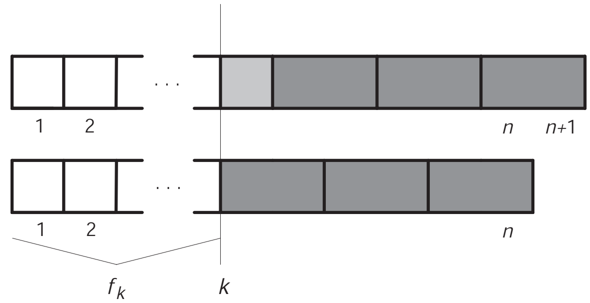

Figure 2: There are ${f_k}^2$ ways to tile these two boards with a fault at cell k.

Since tiling cells 1 through k is independent of tiling after cell k, the number of ways to accomplish both of these tasks is found by multiplying. There are ${f_k}^2$ ways to tile the heads of these two boards and only 1 way to tile the tails, so there are ${f_k}^2 \cdot 1$ tilings of both boards with a fault at cell k. This means that any tiling of such a pair of boards is completely determined by the tiling of their heads. We have done a lot of counting, but we are not quite done. Recall that $0 \leq k \leq n$. Since there cannot be two "last faults" for the boards, one position of k mutually excludes the rest. Therefore, there is a total of $\sum_{k=0}^{n} {f_k}^2$ ways to tile one board of length n and another board of length $n+1$.

Equating the two ways we counted this number completes the proof.

□

3 Visual Example

In order to keep the example small, this visualization uses a board of length 3 above a board of length 2. Note that there are $f_3 f_2 = 3 \cdot 2 = 6$ ways to tile these two boards. All of these ways are listed on the right side of the screen. Then, each pair of tilings is highlighted according to the location of its last fault.

As an example, consider a pair that has a fault at cell 2. The heads of these boards have length 2, and there are only 2 ways to tile a 2-board. In other words, $f_2 = 2$. Consequently, there are ${f_2}^2 = 2^2 = 4$ tilings for a pair of boards with a fault at cell 2.

References

[BQ03] Arthur T. Benjamin and Jennifer J. Quinn. *Proofs that Really Count: The Art of Combinatorial Proof.* Number 27 in The Dolciani Mathematical Expositions. The Mathematical Association of America, 2003.