Introduction to Combinatorial Proofs

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Before you can read, comprehend, or construct combinatorial proofs, you must first understand what they are. I have been told that the word "combinatorial" sounds like something out of a Dr. Suess book, and it's no secret that the word "proof" has intimidated countless undergraduates. The goal of this project is to familiarize you with a variety of combinatorial proofs in the hope that you learn to not only appreciate but also enjoy the art of the combinatorial proof.

Combinatorial proofs come out of a branch of mathematics known as combinatorics, which is the study of arranging and counting elements in sets. These proofs are often used to validate equalities. We think of each side of an equation as one set that can be enumerated. If we can count the same number of elements based on the left-hand side and then in terms of the right-hand side, the equality has been proven.

The hardest part in doing a combinatorial proof is deciding how to count. The first question to ask yourself is, "What is it that we are counting?" To begin, define the set you are working with based on the easier side of the equation. The next task is to determine how the other side counts the same set. While the method sounds simplistic, there is of course more effort involved. It may take a lot of thought and scratch-paper to discover the desired set. There is no algorithm to follow for combinatorial proofs. Usually, there is more than one way to arrive at the solution, and it is this creativity that makes combinatorial proofs so attractive.

Most identities that are proven combinatorially can also be proven in other ways. However, algebraic and induction proofs often lack the elegance of a combinatorial proof. For example, consider the following well-known property of combinations: $\binom{n}{r} = \binom{n}{n-r}$.[1] Although this identity is not difficult to prove using the algebraic definition of a combination, it is more intuitively satisfying to explain it in the words of a combinatorial proof.[2]

Like many mathematical topics, combinatorial proofs can be confusing or overwhelming when they are first presented. In my initial encounter with combinatorial proofs, I did not fully grasp their purpose and dismissed them as math stories. They seemed less rigorous than other types of proofs and therefore less credible. However, I have come to appreciate the ingenuity involved in a combinatorial proof. I discovered that combinatorial proofs often illustrate the meaning of a theorem. As Dr. Benjamin and Dr. Quinn say in their book *Proofs that Really Count: The Art of Combinatorial Proof*, "Counting leads to beautiful, often elementary, and very concrete proofs. While not necessarily the simplest approach, it offers another method to gain understanding of mathematical truths" [BQ03, p.ix].

References

[BQ03] Arthur T. Benjamin and Jennifer J. Quinn. *Proofs that Really Count: The Art of Combinatorial Proof*. Number 27 in The Dolciani Mathematical Expositions. The Mathematical Association of America, 2003.

[Gos03] Eric Gossett. *Discrete Math With Proof*. Prentice Hall, 2003.

[1] The term *combination* is explained in the "Common Definitions" file.

[2] The proof relies on the fact that choosing a subset of size r from a set of size n automatically determines a subset of size $n - r$ consisting of the elements *not* chosen [Gos03].